Abstract. The undefinability in question is proved by constructing a bijection of lines preserving perpendicularity but not intersection.

Introduction

In Schwabhäuser and Szczereba [3] formalizations of Euclidean geometry are considered in which the universe consists of lines and the primitive notions are relations on lines. A set of primitive notions sufficient to formalize n-dimensional Euclidean geometry is found for each $n \geq 2$. For $n \geq 4$, the single binary notion of perpendicularity (two lines intersecting at a right angle) suffices. For $n = 2$, this notion in conjunction with the ternary notion of copunctuality (three lines intersecting at a single point) suffices and is shown to be minimal, in the sense that neither notion is definable from the other. For the remaining case of $n = 3$, the two binary notions of perpendicularity and intersection are shown to be sufficient. It is easy to see that perpendicularity is not definable from intersection. (The proof uses Padoa’s method.) Schwabhäuser and Szczereba ask whether intersection is definable from perpendicularity for $n = 3$. This paper answers that negatively, thus proving that perpendicularity and intersection form a minimal set of primitive notions sufficient to formalize the three-dimensional Euclidean geometry of lines.

Preliminaries

$\mathbb{R}$ is the set of all real numbers, and $\mathbb{R}^3$ is the three-dimensional vector space over $\mathbb{R}$. The operations $\cdot$ and $\times$ are the dot and cross products, respectively. $|a|$ is the length of the vector $a$. In this paper all vectors are in $\mathbb{R}^3$.

A line is any set $A = \{p + \lambda a \mid \lambda \in \mathbb{R}\}$, where $p$ and $a$ are vectors. If $|a| = 1$, then $a$ is called a direction vector of $A$. Every line has exactly two direction vectors. If $a$ is a direction vector, then so is $-a$. Let $\mathcal{L}$ be the set of all lines.

Throughout this paper, real numbers will be denoted by lowercase Greek letters (except $\delta$ and $\theta$), vectors by lowercase Roman letters, and lines by uppercase Roman letters. Given a line denoted by an uppercase Roman


letter, the corresponding lowercase letter will denote one of its direction vectors.

One vector identity which will be used is the following:

$$a \times b \cdot c \times d = (a \cdot c)(b \cdot d) - (a \cdot d)(b \cdot c)$$

The validity of this is easily checked.

Let $p$ lie on $A$ and $q$ lie on $B$, where $A$ and $B$ are not parallel. It is easy to check that

$$A \text{ intersects } B \text{ iff } (p - q) \cdot a \times b = 0.$$ 

A function $\delta: \mathbb{R} \to \mathbb{R}$ is called a derivation provided $\delta(\alpha + \beta) = \delta(\alpha) + \delta(\beta)$ and $\delta(\alpha\beta) = \delta(\alpha)\beta + \alpha\delta(\beta)$ for any $\alpha$ and $\beta$. Note that $\delta$ is not required to be linear over $\mathbb{R}$. If $\delta$ is a derivation and $\alpha$ is algebraic, then $\delta(\alpha) = 0$. (Thus, the product rule implies that $\delta$ is linear over the algebraic reals.) The existence of nontrivial derivations is guaranteed by the following fact.

**FACT.** Let $\alpha$ be any transcendental number. Then there exists a derivation $\delta$ with $\delta(\alpha) \neq 0$.

This is a special case of Lang [2, p. 267].

Given a derivation $\delta$, we define a function $\bar{\delta}: \mathbb{R}^3 \to \mathbb{R}^3$ by $\bar{\delta}(\langle \alpha, \beta, \gamma \rangle) = \langle \delta(\alpha), \delta(\beta), \delta(\gamma) \rangle$. $\bar{\delta}$ satisfies many identities similar to vector calculus identities. Among these are

$$\bar{\delta}(-a) = -\bar{\delta}(a),$$

$$\delta(a \cdot b) = \bar{\delta}(a) \cdot b + a \cdot \bar{\delta}(b),$$

$$|a| = 1 \text{ implies } a \cdot \bar{\delta}(a) = 0,$$

and

$$|a| = 1 \text{ implies } a \times \bar{\delta}(a) \cdot a \times b = \bar{\delta}(a) \cdot b.$$ 

The proofs of (3) and (4) are analogous to the proofs of the corresponding vector calculus identities. For (5), we have

$$a \cdot \bar{\delta}(a) = \frac{1}{2} (\bar{\delta}(a) \cdot a + a \cdot \bar{\delta}(a))$$

$$= \frac{1}{2} \delta(a \cdot a)$$

$$= \frac{1}{2} \delta(1)$$

$$= 0.$$