Smooth Projective Translation Planes

Dedicated to Professor Dr. H. Salzmann on the occasion of his 65th birthday

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Abstract. A projective plane is called smooth if both the point space and the line space are smooth manifolds such that the geometric operations are smooth. We prove that every smooth projective translation plane is isomorphic to one of the classical planes over \( \mathbb{R}, \mathbb{C}, \mathbb{H}, \) or \( \mathbb{O} \).


Introduction

In his fundamental paper of 1971, Breitsprecher ([4]) asked for differentiability conditions characterizing the classical planes over \( \mathbb{R}, \mathbb{C}, \mathbb{H}, \) or \( \mathbb{O} \). Such a characterization needs further assumptions, since there exist non-classical smooth projective planes in each of the possible dimensions 2, 4, 8, 16 (cp. Otte ([14])). Here we consider translation planes.

In the special case of translation planes over division algebras, Grundhöfer and Hähl ([7]) achieved a characterization which works even for affine planes: every smooth plane of Lenz-type V is isomorphic to one of the classical planes. The restriction to a special type of translation planes is essential in the affine case, since again there exist non-classical smooth affine translation planes in each of the possible dimensions \( \{4, 8, 16\} \) (cp. Otte ([14])). In the projective case, we can now omit this restriction by a different approach.

Any line of a smooth projective translation plane may be described as one-point-compactification \( \mathbb{R}^n \cup \{\infty\}, n \in \{1, 2, 4, 8\} \) carrying a smooth structure given by the plane. One may ask which elements of \( \text{GL}(n, \mathbb{R}) \) extend to diffeomorphisms of \( \mathbb{R}^n \cup \{\infty\} \); certain projectivities of the plane have this form. In Section 1 we show that the resulting group \( \text{GL}(n, \mathbb{R}) \cap \text{Diff}(\mathbb{R}^n \cup \{\infty\}) \) of linear diffeomorphisms is comparatively small (cp. 1.6). In Section 2 we adjust this result to our geometric situation and prove that smooth projective translation planes possess small projectivity groups (cp. 2.6). By a theorem of Grundhöfer and Strambach (cp. 2.7), this property characterizes the classical planes among compact, connected translation planes.
1. Linear Diffeomorphisms on Spheres

Let $\mathbb{R}^n \cup \{\infty\}$ be the one-point-compactification of $\mathbb{R}^n$. Its homeomorphism group contains the general linear group $GL(n, \mathbb{R})$ since every $A \in GL(n, \mathbb{R})$ gives rise to a continuous bijection

$$A: \mathbb{R}^n \cup \{\infty\} \to \mathbb{R}^n \cup \{\infty\}; \begin{cases} x \mapsto A(x) & x \neq \infty \\ \infty \mapsto \infty \end{cases}$$

which will also be denoted by $A$. In the following we are interested in the interaction of smooth structures on $\mathbb{R}^n \cup \{\infty\}$ with the resulting groups $GL(n, \mathbb{R}) \cap Diff(\mathbb{R}^n \cup \{\infty\})$ of linear diffeomorphisms.

We want the notation to be consistent with our definition of quasifields (with linear left multiplications) in Section 2, so all vector spaces are considered as right vector spaces.

1.1. The classical smooth structure on $\mathbb{R}^n \cup \{\infty\}$ is defined by the charts $id: \mathbb{R}^n \to \mathbb{R}^n$ and

$$r: (\mathbb{R}^n \cup \{\infty\}) \setminus \{0\} \to \mathbb{R}^n; \begin{cases} x \mapsto x\|x\|^{-2} & x \neq \infty \\ \infty \mapsto 0 \end{cases}$$

The resulting smooth manifold is called the classical sphere of dimension $n$, denoted by $S^n$.

1.2 LEMMA. Let $\Sigma^n = \mathbb{R}^n \cup \{\infty\}$ be a smooth manifold such that the identity map $id: S^n \to \Sigma^n$ is differentiable and regular at $\infty$. Then

$$GL(n, \mathbb{R}) \cap Diff(\Sigma^n) \subseteq GO(n, \mathbb{R}).$$

As an important step in the proof we use an elementary characterization of linear differentiable maps.

1.3 LEMMA. Consider a continuous map $f: \mathbb{R}^n \to \mathbb{R}^n$ which is differentiable in the fixed point $0 \in \mathbb{R}^n$. Then the following are equivalent:

(i) $f$ is a linear map.
(ii) $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}^n$.
(iii) $f(x\xi) = f(x)\xi$ for all $x \in \mathbb{R}^n, \xi \in \mathbb{R}$.

Proof. The implication (i) $\Rightarrow$ (ii) is trivial and (ii) $\Rightarrow$ (iii) is valid for arbitrary continuous maps, as is well known. Concerning (iii) $\Rightarrow$ (i), we note that $f$ coincides with its derivative in 0,

$$Df(0)(x) = \left. \frac{d}{d\xi} f(x\xi) \right|_{\xi=0} \stackrel{(iii)}{=} \left. \frac{d}{d\xi} f(x)\xi \right|_{\xi=0} = f(x),$$