Space Tilings and Substitutions*

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Abstract. We generalize the study of symbolic dynamical systems of finite type and \( \mathbb{Z}^2 \) action, and the associated use of symbolic substitution dynamical systems, to dynamical systems with \( \mathbb{R}^2 \) action. The new systems are associated with tilings of the plane. We generalize the classical technique of the matrix of a substitution to include the geometrical information needed to study tilings, and we utilize rotation invariance to eliminate discrete spectrum. As an example we prove that the pinwheel tilings have no discrete spectrum.

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1. Introduction

Our main objective is to determine the limit of disorder which is possible in a certain class of dynamical systems with \( \mathbb{R}^2 \) action, a class, which we call ‘tiling dynamical systems’, analogous to the symbolic systems of finite type. A subsidiary goal is to extend the classical tools of symbolic substitution dynamics to the geometrical needs of tilings of space.

We begin with a definition of tiling dynamical systems [11], [1]. By a ‘prototile’ \( p \) we mean a homeomorphic image of the closed unit disk in \( \mathbb{R}^2 \), with center of mass at the origin, which has small surface to volume ratio in the sense that:

\[
\frac{\text{area}\{x \in tp: \|x - y\| \leq 1, \text{ for some } y \in \partial(tp)\}}{\text{area}\{tp\}} \rightarrow 0
\]  

(1)

as the expansion factor \( t \to \infty \). Assume given a finite set \( S \) of prototiles, and a subgroup \( G \) (called the ‘allowed isometries’) of the connected subgroup \( \mathcal{E}^2 \) of the topological group of isometries of \( \mathbb{R}^2 \). \( G \) is assumed to be either \( \mathbb{Z}^2 \), \( \mathbb{R}^2 \) or \( \mathcal{E}^2 \), and the image of a prototile by an allowed isometry will be called a ‘tile’; two tiles have the same ‘tile-type’ if they are images of the same prototile. By a ‘swatch’ we will mean a finite collection of tiles with pairwise disjoint interiors. A final requirement: we assume that the boundary of a tile can be covered by tiles, all with pairwise disjoint interiors, in only finitely many ways up to isometry. We associate with given \( S \) and \( G \) the set \( X(S) \) of all infinite collections of tiles, called ‘tilings’, that

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have union $\mathbb{R}^2$, and are such that each pair of tiles in a tiling has disjoint interiors. We assume $X(S)$ is nonempty.

We next define a topology on $X(S)$. There is an obvious action of $G$ on the space $X(S)$. Then a typical element of a base for the topology on $X(S)$ is $C(P, \epsilon)$, where $P$ is a swatch $\{r_j[p_j(p_{ij})] : 1 \leq j \leq n\}$, ($p_j$ being a prototile, $r_j$ an allowed rotation about the origin, and $r_j$ an allowed translation), consisting of all tilings which contain as a swatch $P' = \{r'_j[p'_j(p_{ij})] : 1 \leq j \leq n\}$ where $\|r_j - r'_j\| < \epsilon/2$ and $\|p_j - p'_{ij}\| < \epsilon/2$. It is not hard to check that with this topology $X(S)$ is metrizable and compact, and that $G$ acts continuously on it [11].

Let $G_T$ be the subgroup of translations in $G$. The ‘tiling dynamical system’ associated with given $S$ and $G$ is $(X(S), G_T)$. With this notation, our main objective is to determine the limits of disorder possible for uniquely ergodic tiling dynamical systems; in particular we wish to determine if such a system can have pure absolutely continuous spectrum. We do not solve this problem, but we do make a significant generalization of classical substitution dynamics and give evidence of its usefulness in our problem. Our basic problem has origins in several directions. For a review see [7], and for more recent developments see [8]-[11], [1].

One historical root of our problem was the search for tiling dynamical systems which have no closed orbits. The original examples were obtained by use of some form of hierarchical structure; we will follow this path. So given a tiling dynamical system, we assume further that we are given a ‘substitution function’ $F$ on the set of prototiles, which applied to prototile $p$ produces a finite collection $F(p)$ of tiles, with pairwise disjoint interiors, such that the set theoretic union (also denoted $F(p)$) is geometrically similar to $p$. We call the collection, or any allowed isometry of it, an ‘LS–tile’. LS–tiles $F(p)$ must have the same expansion factor $E > 1$ for all tiles $p$. One can iterate the substitution, producing larger and larger collections of tiles, each collection remaining geometrically similar to its predecessor. We will call any such collection, or an allowed isometry of such a collection, a ‘VLS–tile’, and define its ‘order’ as the number of substitutions used to produce it from a tile. (So an LS–tile is a VLS–tile of order 1.)

Given the substitution $F$ we define the closed, $G$-invariant subset $X(S)_F$ of $X(S)$ to consist of all tilings in which each swatch is a subset of a VLS–tile. (We call $(X(S)_F, G_T)$ a ‘substitution-tiling dynamical system’.) It can easily be proven that every tiling in $X(S)_F$ decomposes (not necessarily uniquely) into VLS–tiles of any fixed order with disjoint interiors. (Consider an expanding sequence of finite subcollections of a tiling, namely all the tiles inside an expanding disk. Using the definition we see then that every tiling is the union of an expanding sequence of VLS–tiles, and by diagonalization we can assume that in this expanding sequence the grouping of tiles into VLS–tiles of any one fixed order does not change.) We also note [11] that for all small enough $\epsilon$, swatches $P'$ associated with open sets $C(P, \epsilon)$ must be of the form $P' = g(P)$ for some $g \in G$.

Previous examples, of tiling dynamical systems without closed orbits, were obtained by use of a substitution function $F$ (or something similar) such that