MAXIMALITY OF $\text{PSp}_6(q)$ ACTING ON THREE-DIMENSIONAL COMPLETELY ISOTROPIC SUBSPACES

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UDC 512.542.7

Let $G$ be a finite simple Lie group, let $M$ be a class of conjugate maximal parabolic subgroups of $G$, and let $(G, M)$ be a substitution group that corresponds to the action of $G$ with the aid of conjugates on $M$. It is conjectured [1] that the normalizer of a group $G$ is usually a maximal subgroup in the symmetric group $S(M)$ in the alternating group $A(M)$. The first result supporting this conjecture was a theorem on the maximality of $\text{PGL}_n(q)$, $n \geq 3$, acting on straight lines [2]. The proof of the subsequent results [3, 4] is based on this theorem for $n = 3$. Further advances in this respect are related to a study of groups of small rank.

**Theorem.** Let $(\text{PSp}_6(q), N_3)$ be a substitution group corresponding to the natural action of $\text{PSp}_6(q)$, $q > 3$, $q \neq 2^h$ on a set $N_3$ of three-dimensional completely isotropic subspaces in $F_q^6$. Then any substitution group on $N_3$ that contains $\text{PSp}_6(q)$ will either be contained in $(\text{PSp}_6(q), N_3)$, or it will contain the alternating group $A(N_3)$.

This theorem remains valid also for $q = 3$. The proof known to the authors differs from the proof given below.

1. Let $(G, W)$ be a substitution group. A stabilizer of the set $M \subseteq W$ is defined by the subgroup $G_{(M)} = \{g \in G \mid M^g = M\}$. If $M$ is a transitivity block of $G$, i.e., $G_{(M)} = G$, then there exists a natural homomorphism $\lambda : G \rightarrow S(M)$ of the group $G$ into a symmetric group of the set $M$ that is called the action of $G$ on the set $M$. $\text{Ker} \lambda = G_{(M)} = \{g \in G \mid x^g = x, x \in M\}$ is the fixing element of $M$. If $\lambda$ is injective, i.e., $\text{Ker} \lambda = G_{(M)} = 1$, then the action is said to be exact. We shall say that the action $\lambda$ is similar to the substitution group $(G/\text{Ker} \lambda, M)$.

A group $(G, W)$ is said to be regular if $G$ is transitive and the stabilizer of the point $Gx$ is equal to the unit subgroup. In the same way we shall say that an action is regular if it is similar to a regular substitution group. An Abelian group acts regularly on any of its orbits. The centralizer of a regular substitution group $(G, W)$ in $S(W)$ is also regular.

The group $G$ acts naturally on the set $\hat{W}^k = \{(a_1, ..., a_k) \mid a_i \in W, a_i \neq a_j, i \neq j\}$. The elements of $\hat{W}^k$ are called $k$-points. The transitivity blocks and the orbits of $(G, \hat{W}^k)$ are called invariant $k$-relations and invariant $k$-orbits of $(G, W)$. A relation $\theta$ is said to be symmetrical if together with each $k$-point $(a_1, ..., a_k)$ it contains all the $k$-points obtained from $(a_1, ..., a_k)$ by permutation of coordinates. Inclusion-minimal symmetrical invariant $k$-relations are called symmetrized $k$-orbits. Let $\phi$ be a $(k + 1)$-nary invariant relation, and let $O$ be a $k$-orbit of the group $G$. The number $\phi(a_1, ..., a_k, y) = |\{(a_1, ..., a_k, y) \in O\}|$ does not depend on the selection of a $k$-point $(a_1, ..., a_k) \in O$ and it is called the coefficient of projection of $\phi$ on the $k$-orbit $O$.

2. Let us study some properties of the group $\text{PSp}_6(q)$. We shall adopt the notation used in [5]. Let $F_q$ be a field of $q$ elements, $q = p^h$, $p > 2$. We shall consider a six-dimensional vector space $E$ over $F_q$ and an $f$-alternating bilinear form on $E$. By $N_3$ we shall denote the set of maximal (three-dimensional) completely isotropic subspaces of $E$. The substitution group $(\text{PSp}_6(q), N_3)$ is primitive and it has the following binary orbits: $\Gamma_i = \{(V, W) \in N_3^2 \mid \dim V \cap W = 3 - i, i = 1, 2, 3, \Gamma_i \}$ is called a neighbor relation.


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In $E$ let us select a symplectic basis $\{n_1, n_2, n_3, m_1, m_2, m_3\}$. The group $Sp_6(q)$ can be regarded as a matrix group $\{A \in GL_6(q) | ATA^t = T\}$, where $T = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, and $I$ is a unit $(3 \times 3)$-matrix. The elements of $PSp_6(q)$ will be represented by the matrix of the corresponding linear transformation belonging to $Sp_6(q)$. We can assign an element of $N_3$ by indicating the coordinates of its basis.

Let us find the 3-orbits of the group $(PSp_6(q), N_3)$. The relations $\Delta(k, l, m, n) = \{X, Y, Z \in (N_3) | \dim X \cap Y \cap Z = k, \dim X \cap Y = k + \ell, \dim X \cap Z = k + m, \dim Y \cap Z = k + n\}$ are invariants. By $\Delta(k, l, m, n)$ we denote a symmetrization of the relation $\Delta(k, l, m, n)$: $\Delta(k, l, m, n) = \Delta(k, l, m, n) \circ \Delta(k, l, m, n) \cup \Delta(k, l, m, n)$.

**Lemma 1.** There exist 15 symmetrized 3-orbits $\Phi_1, \ldots, \Phi_{15}$ of the group $(PSp_6(q), N_3): D_1 = \Delta(0, 2, 0, 0), \Phi_2 \cup \Phi_3 = \Delta(0, 1, 1, 0), \Phi_4 \cup \Phi_5 = \Delta(0, 1, 0, 0), \Phi_6 \cup \Phi_7 = \Delta(0, 0, 0, 0), \Phi_8 = \Delta(0, 0, 0, 0), \Phi_9 = \Delta(0, 1, 0, 0), \Phi_{10} = \Delta(1, 0, 0, 0), \Phi_{11} \cup \Phi_{12} = \Delta(1, 1, 1, 0), \Phi_{13} = \Delta(1, 1, 0, 0), \Phi_{14} = \Delta(2, 0, 0, 0), \Phi_{15} = \Delta(2, 0, 0, 0)$. The nonzero projection coefficients $\alpha_i = q$ can be calculated by the formulas $\alpha_1 = 2q^4, \alpha_2 = 2q^4, \alpha_3 = q^2(q^2 - 1)/2, \alpha_4 = q^2(q^2 - 1)/2, \alpha_5 = q^2(q^2 - 1)/2, \alpha_6 = q^2(q^2 - 1)/2, \alpha_7 = q^2(q^2 - 1)/2, \alpha_8 = q^2(q^2 - 1)/2, \alpha_9 = q^2(q^2 - 1)/2, \alpha_{10} = q^2(q^2 - 1)/2, \alpha_{11} = q^2(q^2 - 1)/2, \alpha_{12} = q^2(q^2 - 1)/2, \alpha_{13} = q^2(q^2 - 1)/2, \alpha_{14} = q^2(q^2 - 1)/2, \alpha_{15} = q^2(q^2 - 1)/2$.

Proof. Let $(V, W) \in \Gamma_1$. The number of 3-orbits in $\Delta(k, l, m, n)$ is equal to the number of orbits of $PSp_6(q)$ on the set $\{X \in N_3 | \dim X \cap Y \cap Z = k, \dim X \cap Y = k + \ell, \dim X \cap Z = k + m, \dim Y \cap Z = k + n\}$. Thus we can find out into how many symmetrized 3-orbits each of the relations $\Delta(k, l, m, n)$ decomposes.

Let $(V, W, X) \in \Phi_j, (V, W) \in \Gamma_1$. The projection coefficient $\alpha_j$ of the relation $\Delta(k, l, m, n)$ on $\Phi_j$ is equal to $[PSp_6(q) : PSp_6(q) \cap (V, W, X)]$ if $(V, W, X)$ and $(W, V, X)$ lie in the same 3-orbit of $PSp_6(q)$, and to $2[PSp_6(q) : PSp_6(q) \cap (V, W, X)]$ otherwise.

As an example let us consider $\Delta(0, 1, 0, 0)$. Let us write $V = \langle n_1, n_2, n_3 \rangle$ and $W = \langle m_1, m_2, m_3 \rangle$. The substitution $h \in PSp_6(q)_{vw}$ is specified by a matrix $\begin{pmatrix} A^t & 0 \\ B & A^{-1} \end{pmatrix}$, where $A \in GL_3(q)$ and $B$ is a symmetric matrix. If $h$ specifies $W$, then it is easy to see that $A = \begin{pmatrix} * & 0 & 0 \\ 0 & A_t \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, $|PSp_6(q)_{vw}| = q^3(q - 1)^3(q + 1)$. Let $X \in N_3, X \cap V = 0$. Then we can (uniquely) select in $X$ a basis $l_i = m_1 + \beta_{i1}n_1 + \beta_{i2}n_2 + \beta_{i3}n_3, i = 1, 2, 3$, with the matrix $C = (\beta_{ij})$ being symmetrical [since $f(l_i, l_j) = 0$]. By requiring that $X \cap W = 0$, we obtain $[\beta_{12}, \beta_{22}, \beta_{32}] = 0$. The substitution $h$ carries $X$ into $X' = \langle m_1 + \beta_{i1}n_1 + \beta_{i2}n_2 + \beta_{i3}n_3, i = 1, 2, 3 \rangle, C' = A(B + C)A^t$. The matrix $B$ can be selected in such a way that $B + C = \begin{pmatrix} 0 & 0 \\ 0 & C_1 \end{pmatrix}$, then $C' = \begin{pmatrix} 0 & 0 \\ 0 & C_1 \end{pmatrix}, C_i = A_1C_iA_1^t$, i.e., $C_1$ varies as a matrix of a bilinear form. It is well known that over $F_q$, $\text{char } F_q \neq 2$ there exist two classes of nonsingular symmetric bilinear forms, with $D_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $D_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ being the matrices of their representatives (g is a quadratic mismatch of $F_q$). Hence with an appropriate choice of $h \in PSp_6(q)_{vw}$, $\alpha_2 = \alpha_3 = q^3(q - 1)^3(q + 1)/2 = q^3(q^2 - 1)/2$, $\alpha_7 = q^3(q - 1)^3(q + 1)/2 = q^3(q^2 - 1)/2$, $\alpha_9 = q^3(q - 1)^3(q + 1)/2 = q^3(q^2 - 1)/2$.

Similarly, $\alpha_3 = 2[PSp_6(q)_{vw} : PSp_6(q)_{vw}] = q^3(q^2 - 1)(q^2 - 1)/2$.