A LOWER BOUND FOR RANDOMIZED ALGEBRAIC DECISION TREES

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Dedicated to the memory of Roman Smolensky

Abstract. We prove the first nontrivial (and superlinear) lower bounds on the depth of randomized algebraic decision trees (with two-sided error) for problems being finite unions of hyperplanes and intersections of halfspaces, solving a long standing open problem. As an application, among other things, we derive, for the first time, an $\Omega(n^2)$ randomized lower bound for the Knapsack Problem, and an $\Omega(n \log n)$ randomized lower bound for the Element Distinctness Problem which were previously known only for deterministic algebraic decision trees. It is worth noting that for the languages being finite unions of hyperplanes our proof method yields also a new elementary lower bound technique for deterministic algebraic decision trees without making use of Milnor's bound on Betti number of algebraic varieties.

Key words. Computational Complexity; Randomized Algebraic Decision Trees; Knapsack; Element Distinctness; Integer Programming.
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1. Introduction

Starting with Manber & Tompa (1985), Snir (1985), Meyer auf der Heide (1985a) and Meyer auf der Heide (1985b) there has been a continued effort in the last decade to understand the intrinsic power of randomization in algebraic decision trees (see also Bürgisser, Karpinski & Lickteig (1993), Grigoriev & Karpinski (1993), Grigoriev & Karpinski (1994) for some more recent results). Several algebraic and topological methods which were introduced in proving lower bounds for deterministic algebraic decision trees (cf. Yao (1981), Steele & Yao (1982), Ben-Or (1983), Björner, Lovász & Yao (1992), Yao (1992), Grigoriev, Karpinski & Vorobjov (1997), Yao (1994)), with the exception of...
Bürgisser, Karpinski & Lickteig (1993), and Grigoriev & Karpinski (1993), were not yielding lower bounds for the case of randomized decision trees. In Meyer auf der Heide (1985a) a lower bound has been stated on the depth of randomized linear decision trees (randomized algebraic decision trees of degree 1) for the case of languages being finite unions of hyperplanes (a gap in the proof of the Main Lemma of Meyer auf der Heide (1985a) for the generic case was closed in Grigoriev & Karpinski (1994)). Our paper provides the first lower bounds on the depth of randomized algebraic decision trees in the case of the problems being finite unions of hyperplanes as well as intersections of halfspaces. This provides also a new method for proving lower bounds for deterministic algebraic decision trees without making use of Milnor's bound and Betti numbers of algebraic varieties. As an application we derive randomized lower bounds for a number of concrete problems, among others, \textit{Knapsack} ($\Omega(n^2)$ lower bound), and the \textit{Element Distinctness} ($\Omega(n \log n)$ lower bound).

The paper is organized as follows. Section 2 introduces randomized algebraic decision and computation trees. Section 3 overviews the known results in the area. Section 4 summarizes our results and applies them for a number of concrete problems. Section 5 gives an outline of the lower bound proof, and Sections 6 and 7 give the proof of the Main Theorem. Section 8 contains the complexity lower bound for deterministic decision trees under less restrictive conditions than in the Main Theorem for their randomized counterparts.

2. Deterministic and randomized decision trees

An algebraic decision tree of degree $d$, a $d$-DT, for inputs $(x_1, \ldots, x_n) \in \mathbb{R}^n$ is a rooted ternary tree. Its root and inner nodes are labelled by polynomials from $\mathbb{R}[X_1, \ldots, X_n]$ of degree at most $d$, its leaves are accepting or rejecting. The computation of the $d$-DT on input $(x_1, \ldots, x_n) \in \mathbb{R}^n$ consists of traversing the tree from the root to a leaf, always choosing the left/middle/right branch of a node labelled with polynomial $g$ depending on whether $g(x_1, \ldots, x_n)$ is smaller/equal/greater than 0.

The inputs $(x_1, \ldots, x_n) \in \mathbb{R}^n$ arriving at accepting leaves form the set $S \subseteq \mathbb{R}^n$ recognized by the $d$-DT.

We deal in this paper with randomized algebraic decision trees of degree $d$, d-RDTs for short. There are several variants of this model known in the literature (see, e.g. Meyer auf der Heide (1985a), Meyer auf der Heide (1985b), Meyer auf der Heide (1985c), Manber & Tompa (1985)). One of the most natural variants allows coin flipping nodes and charges for the random bits used. Other variants ignore the costs for random choices, or pull flipping nodes out of the