EDGE PARTITIONS OF THE RADO GRAPH

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We will prove that for every coloring of the edges of the Rado graph, \( \mathcal{R} \) (that is the countable homogeneous graph), with finitely many colors, it contains an isomorphic copy whose edges are colored with at most two of the colors. It was known \([4]\) that there need not be a copy whose edges are colored with only one of the colors. For the proof we use the lexicographical order on the vertices of the Rado graph, defined by Erdős, Hajnal and Pósa.

Using the result we are able to describe a “Ramsey basis” for the class of Rado graphs whose edges are colored with at most a finite number, \( r \), of colors. This answers an old question of M. Pouzet.

1. Notation

All graphs \( \mathcal{G} \) under consideration will consist of a set \( V(\mathcal{G}) = G \) of vertices together with a symmetric and reflexive binary relation \( \sim \). If \( x \sim y \) we will say that the vertex \( x \) is adjacent to the vertex \( y \). If the vertex \( x \) is not adjacent to the vertex \( y \) we write \( x \not\sim y \). The set of pairs \( \{x, y\} \) such that \( x \sim y \) and \( x \not\sim y \) is the set of edges of \( \mathcal{G} \). Note that \( \{x, x\} \) is not an edge of \( \mathcal{G} \) although \( x \sim x \) for all \( x \in G \). For \( a \in G \) the set of neighbors of \( a \) is denoted by \( \Gamma(a) = \{b \in G : a \sim b \land a \not\sim b\} \).

The Rado graph \( \mathcal{R} = (\omega; \sim) \) is the unique countable graph with the property that for every finite graph \( \mathcal{G} = (G; \sim) \), vertex \( a \in G \) and embedding \( \alpha : \mathcal{G} - a \rightarrow \mathcal{R} \) there is an extension of the embedding \( \alpha \) to an embedding \( \alpha' : \mathcal{G} \rightarrow \mathcal{R} \). (A good source for facts on the Rado graph is \([1]\).) This defining property of the Rado graph is called the mapping extension property; it implies that if \( F \) is a finite subset of \( \omega \) and \( S \subseteq F \) then there are infinitely many vertices \( x \in \omega \) such that \( x \) is adjacent to every vertex in \( S \) and not adjacent to any vertex in \( F - S \).

Let \( \mathcal{R} = (\omega; \sim) \) be the Rado graph. A pair \( (F; x) \) with \( F \cup \{x\} \subseteq \omega, x \not\in F \) and \( F \) finite is a type of \( \mathcal{R} \), and the orbit of the type \( (F; x) \) is

\[
\text{Orb}(F; x) = \{y \in \omega - F : \forall z \in F \ (z \sim x \Leftrightarrow z \sim y)\}.
\]

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Observe that that $x \in \text{Orb}(F; x)$ and $\text{Orb}(\emptyset; x) = \omega$. It follows from the definition that

$$\forall y \in \text{Orb}(F; x) \quad (\text{Orb}(F; x) = \text{Orb}(F; y)).$$

Note that for $a \in \omega$ we have $\text{Orb}({a}; x) = \Gamma(a)$ or $R - (\Gamma(a) \cup \{a\})$ according as $a \sim x$ or $a \nfall x$. If the vertices $y$ and $z$ are both elements of $\text{Orb}(F; x)$ we say that $y$ and $z$ are of the same type over $F$. The types $(F; x)$ and $(G; y)$ are disjoint if $F \cap G = \emptyset$.

Let $\mathfrak{G}$ be a countable graph and $a$ a vertex of $\mathfrak{G}$ and $\sigma = (\sigma_0, \sigma_1, \sigma_2, \ldots)$ a finite or infinite sequence of vertices of $\mathfrak{G}$. We define $\text{lex}(a, \sigma)$, the lexicographic value of the vertex $a$ with respect to the sequence $\sigma$. If the vertex $a$ is an element of the sequence $\sigma$ and $j$ is the smallest index such that $a = \sigma_j$ then $\text{lex}(a, \sigma)$ is the sequence $(\epsilon_i : i \leq j)$ such that

$$\epsilon_i = \begin{cases} 1 & \text{if } a \sim \sigma_i, \\ 0 & \text{if } a \nfall \sigma_i. \end{cases}$$

Note that the sequence $\text{lex}(a, \sigma)$ has a “1” in the $j$-th place because $a$ is adjacent to $a$. If the vertex $a$ is not an element of the sequence $\sigma$ let $\sigma'$ be the sequence obtained from the sequence $\sigma$ by adding the vertex $a$ as its last element. (We will use this only when $\sigma$ is a finite sequence; although, the definition has an obvious interpretation for sequences of any order type.) Then $\text{lex}(a, \sigma) = \text{lex}(a, \sigma')$.

For sequences $\sigma = (\sigma_i : i \in \nu)$ and $\tau = (\tau_i : i \in \nu)$, we write $\sigma \leq \tau$ if and only if $\sigma$ lexicographically precedes $\tau$. The sequence $(\sigma_i : i \in \nu)$ is an extension of the sequence $(\sigma_i : i \in \mu)$ if the ordinal $\mu$ is less than or equal to the ordinal $\nu$. The sequence $(\delta_i : i \in \mu)$ is a subsequence of $(\sigma_i : i \in \nu)$ if there is an order preserving injection $\alpha$ from $\mu$ into $\nu$ such that $\forall i \in \mu \ (\delta_i = \sigma_{\alpha(i)})$. We will only use sequences $(\sigma_i : i \in \nu)$ of distinct vertices of $\mathfrak{G}$, $\forall i, j \in \nu \ (i \neq j \rightarrow \sigma_i \neq \sigma_j)$. The function $\sigma^{-1}$ is then well defined on the range of $\sigma$. It follows easily from the definitions that if a sequence $\sigma'$ of vertices of $\mathfrak{G}$ is an extension of the sequence $\sigma$ and $\text{lex}(a, \sigma) \prec \text{lex}(b, \sigma)$ then $\text{lex}(a, \sigma') \prec \text{lex}(b, \sigma')$. We will also need the following consequence of the definitions.

Let $\sigma = (\sigma_i : i < n)$ be a finite sequence of vertices of $\mathfrak{G}$ and let $\sigma' = (\sigma_i : i \leq n)$ be an extension of $\sigma$ with $\sigma_n$ a vertex of $\mathfrak{G}$. If $a \in G$ is not an element of $\sigma'$ and $\text{lex}(a, \sigma) = \text{lex}(\sigma_n, \sigma)$, then $a \nfall \sigma_n$ implies that $\text{lex}(a, \sigma') \prec \text{lex}(\sigma_n, \sigma')$ and $a \sim \sigma_n$ implies that $\text{lex}(a, \sigma') \succ \text{lex}(\sigma_n, \sigma')$.

Let $\mathcal{R} = (\omega; \sim)$ be the Rado graph and let $\sigma$ be any permutation of $\omega$. Thus $(\sigma_0, \sigma_1, \ldots)$ is an enumeration of the vertices of $\mathcal{R}$. Define an order relation $\leq_{\sigma}$ on $\omega$ by

$$a \leq_{\sigma} b \iff \text{lex}(a, \sigma) \leq \text{lex}(b, \sigma),$$

We call $\leq_{\sigma}$ a lexicographic order, and denote the structure $(\omega; \sigma, \sim, \prec_{\sigma})$ by $\mathcal{R}_{\sigma}$. $\mathcal{R}_{\sigma}$ is called a lexicographically ordered Rado graph. The order $\prec_{\sigma}$ defines a partition of the set of edges of $\mathcal{R}$, $E = E(\prec_{\sigma}) \cup E(\succ_{\sigma})$, where

$$E(\prec_{\sigma}) = \{\{\sigma_i, \sigma_j\} : i < j \land \sigma_i \sim \sigma_j \land \sigma_i <_{\sigma} \sigma_j\},$$

$$E(\succ_{\sigma}) = \{\{\sigma_i, \sigma_j\} : i < j \land \sigma_i \sim \sigma_j \land \sigma_i >_{\sigma} \sigma_j\}.$$