THE GALLAI–YOUNGER CONJECTURE FOR PLANAR GRAPHS

B. A. REED and F. B. SHEPHERD

Received April 11, 1995

Younger conjectured that for every $k$ there is a $g(k)$ such that any digraph $G$ without $k$ vertex disjoint cycles contains a set $X$ of at most $g(k)$ vertices such that $G - X$ has no directed cycles. Gallai had previously conjectured this result for $k = 1$. We prove this conjecture for planar digraphs. Specifically, we show that if $G$ is a planar digraph without $k$ vertex disjoint directed cycles, then $G$ contains a set of at most $O(k \log(k) \log(\log(k)))$ vertices whose removal leaves an acyclic digraph. The work also suggests a conjecture concerning an extension of Vizing’s Theorem for planar graphs*.

1. Introduction and overview

A cycle packing in a digraph $G$ is a set of vertex disjoint directed cycles of $G$. A cycle cover of a digraph $G$ is a set of vertices $X$ such that $G - X$ contains no directed cycles, i.e. $G - X$ is acyclic. The cycle packing number of $G$, denoted $cp(G)$, is the maximum cardinality of a cycle packing in $G$. The cycle cover number of $G$, denoted $cc(G)$, is the minimum cardinality of a cycle cover of $G$. Obviously, $cp(G) \leq cc(G)$. Gallai [1] conjectured that there is a constant $K$ such that $cc(G) \leq K$ for any graph $G$ with $cp(G) = 1$. The analogous question for undirected graphs was resolved by Bollobás [2]. Younger [9] conjectured that there is some function $g$ such that $cc(G) \leq g(cp(G))$. We refer to this as the Gallai–Younger Conjecture and note that it is the directed analogue of the earlier Erdős–Pósa Theorem. Note that the Gallai–Younger Conjecture is equivalent to the statement that there is a function $h$, going to infinity with $n$, such that every digraph $G$ contains $h(cc(G))$ vertex disjoint directed cycles.

Mathematics Subject Classification (1991): 05C75, 05C70, 90C10, 90C27

* Since the time of submission the Gallai-Younger conjecture has been resolved for general graphs by Reed, Robertson, Seymour and Thomas. The bound given for general graphs is worse than exponential. Recently, Goemans and Williamson (IPCO, Vancouver, B.C., June 1996) have given a constant error factor between the parameters $cc(G), fcc(G)$ in the case of planar directed graphs $G$. Together with the results in this paper, this shows that $g(k)$ can be chosen as $Ck$ for some constant $C$ in the planar case.
Throughout the paper, all cycles are directed.

McCuaig [4] proved that a digraph with no two vertex disjoint cycles contains a cycle cover consisting of at most three vertices, thus settling the conjecture of Gallai. Hence the Gallai–Younger conjecture holds for graphs with cp(G) = 1.

Thomassen [7] studied digraphs with large minimal outdegree (the outdegree of a vertex in a simple digraph is the number of arcs leaving it, the minimal outdegree of G, denoted δ+(G), is the minimum of the outdegrees of its vertices). For any acyclic G we have δ+(G) = 0, thus cc(G) ≥ δ+(G). Thomassen proved that there is a function f such that any simple digraph G with δ+(G) ≥ f(k) contains at least k vertex disjoint cycles (for a related conjecture, see [3]).

Metzlar and Murty [5] proved the Gallai–Younger conjecture for planar digraphs with maximum indegree and outdegree 2. Specifically, they showed that for any such graph cc(G) < 4cp(G).

Seymour [6] pointed out a relationship between the cycle cover number and the related fractional cycle cover number. A fractional α cycle cover is a weight function w from V to [0,1] such that:

\[ \sum_{v \in V} w(v) = \alpha \]

and each cycle C satisfies:

\[ \sum_{v \in C} w(v) \geq 1. \]

A cycle cover is simply a fractional cycle cover in which each weight is 0 or 1. The fractional cycle cover number of G, denoted fcc(G), is the smallest α for which G has a fractional α cycle cover. Clearly, fcc(G) ≤ cc(G). Note also that fcc(G) is the optimal value for the linear program (LP)

\[ \text{min}\{1^T w : Aw \geq 1, w \geq 0\} \]

where A is the cycle-vertex incidence matrix for G. Seymour proved that

\[ cc(G) = O(fcc(G) \log(fcc(G)) \log(\log(fcc(G)))) \].

We are going to apply Seymour’s result to prove the Gallai–Younger conjecture for planar digraphs. To do so, we need to introduce fractional cycle packings. A fractional α cycle packing in G is a weight function w from the set of cycles of G to [0,1] such that:

\[ \sum_{C \text{ a cycle of } G} w(C) = \alpha, \]

and for every vertex v in G

\[ \sum_{C \ni v} w(C) \leq 1. \]