THE K–THEORY AND THE INVERTIBILITY OF ALMOST PERIODIC TOEPLITZ OPERATORS

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We use K–theory to study the invertibility of systems of almost periodic Toeplitz operators.

INTRODUCTION

Let Π be an additive of subgroup R. Let AP(Π) be the collection of almost periodic functions f on R with a Fourier series \( f(t) = \sum_{\lambda \in \Gamma} a_{\lambda} e^{2\pi i \lambda t} \). It is well known that \( AP(\Pi) \cong C(\hat{\Pi}) \), where \( \hat{\Pi} \) is the Bohr compactification of \( \Pi \) [16]. Let \( H^2(\mathbb{R}) \) be the Hardy space of the upper half–plane and let \( P: L^2(\mathbb{R}) \to H^2(\mathbb{R}) \) be the orthogonal projection. The C*-algebra generated by all the Toeplitz operators \( T_{\varphi} = PM_{\varphi}|H^2(\mathbb{R}), \varphi \in AP(\Pi) \cong C(\hat{\Pi}) \), will be denoted by \( T(\Pi) \).

The commutator ideal of \( T(\Pi) \), i.e. the ideal generated by \( AB - BA, A, B \in T(\Pi) \), will be denoted by \( C(T(\Pi)) \). The algebras \( T(\Pi) \) and \( C(T(\Pi)) \) have been well studied by many authors [2, 3, 4, 5, 9, 12, 15, 18].

When \( \Pi \) is dense in \( \mathbb{R} \), there is a Breuer–Fredholm index theory related to a type–II \( \infty \) von Neumann algebra for Toeplitz operators \( T_{\varphi} \in T(\Pi) \) [3, 5, 9, 15]. Index is

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generally viewed as an obstruction to inverting an operator; any
study of index theory is always related to the investigation of
the invertibility of the operators in question. The index
theory for $\mathcal{T}(\Gamma)$ tells us, among other things, that for a
$\varphi \in \text{AP}(\Gamma) \subseteq \text{C}(\hat{\Gamma})$, $T\varphi$ is invertible if and only if $\varphi$ is
invertible and the topological index of $\varphi$, which is none other
than the mean motion of the function, is 0 [3,5]. In the view
point of K-theory, this means $[\varphi] = 0$ in $K_1(\text{C}(\hat{\Gamma}))$.

Such a result would naturally lead one to investigate
the invertibility of Toeplitz operator $T_{\Phi}$ on $H^2(\mathbb{R}) \otimes \mathbb{C}^{n}$ with
a matrix symbol $\Phi \in C(\hat{\Gamma}) \otimes M_n$, where $M_n$ is the collection of
$n \times n$ matrices. While the Breuer-Fredholm index theory can be
generalized to this class of operators [5,15], the topological
index of $\Phi$, as one can see from simple examples, is no longer
the only obstruction to invertibility. What, then, are the
other obstructions?

In search for an answer, one turns to the K-theory
exact sequence
\[ K_0(\text{C}(\Gamma)) \rightarrow K_0(\mathcal{T}(\Gamma)) \rightarrow K_0(\text{C}(\hat{\Gamma})) \]
\[ K_1(\text{C}(\hat{\Gamma})) \rightarrow K_1(\mathcal{T}(\Gamma)) \rightarrow K_1(\text{C}(\Gamma)) \]
which comes from the well known short exact sequence
\[ 0 \rightarrow \text{C}(\Gamma) \rightarrow \mathcal{T}(\Gamma) \rightarrow \text{C}(\hat{\Gamma}) \rightarrow 0 \] [3,5,9],
hoping that $K_* (\mathcal{T}(\Gamma))$ might shed some light on the
invertibility of systems of Toeplitz operators. In a joint work
[12] by R. Ji and the author, the K-groups of the commutator
ideal $\text{C}(\Gamma)$ were computed for the purpose of classifying this
class of simple $C^*$-algebras. But this computation was not done
in the context of (0.1). Rather, it was done by making
connection between $\text{C}(\Gamma)$ and the $C^*$-algebra $C_0(\mathbb{R}) \times \Gamma$. It was