SOME COMPUTATIONAL PROBLEMS IN LINEAR ALGEBRA AS HARD AS MATRIX MULTIPLICATION

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Abstract. We define the complexity of a computational problem given by a relation using the model of computation trees together with the Ostrowski complexity measure. Natural examples from linear algebra are:

○ \( \text{KER}_n \): Compute a basis of the kernel for a given \( n \times n \)-matrix,

○ \( \text{OGB}_n \): Find an invertible matrix that transforms a given symmetric \( n \times n \)-matrix (quadratic form) into diagonal form,

○ \( \text{SPR}_n \): Find a sparse representation of a given \( n \times n \)-matrix.

To such a sequence of problems we assign an exponent, similarly as for matrix multiplication. For the complexity of the above problems we prove relative lower bounds of the form \( aM_n - b \) and absolute lower bounds \( dn^2 \) where \( M_n \) denotes the complexity of matrix multiplication and \( a, b, d \) are suitably chosen constants. We show that the exponents of the problem sequences \( \text{KER}, \text{OGB}, \text{SPR} \) are the same as the exponent \( \omega \) of matrix multiplication.

Key words. Problems, computation trees, straight line programs, Ostrowski complexity, derivations, matrix multiplication.

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1. Introduction

It is well known that matrix multiplication is crucial for many computational problems in linear algebra. Problems like matrix inversion, computation of the determinant or of all coefficients of the characteristic polynomial, LR-decomposition, and over the complex numbers also QR-decomposition and unitary transformation to Hessenberg form, are all known to be as hard as matrix multiplication.(See [3], [5], [8], [9], [13], [14], [17].) In this paper we study some
computational problems in linear algebra that are specified more generally by a relation rather than by a function.

Let $F$ denote a field of characteristic zero or an ordered field. The reader may keep in mind the two important examples $F = \mathbb{C}$ or $F = \mathbb{R}$. A problem is given by a relation

$$\Pi \subseteq F^m \times F^m.$$  

(A relational connection between inputs and outputs is the natural way computational problems are specified; see also [4], [12].) Given an input $x \in F^m$ we are asked to find a $y \in F^m$ such that $(x, y) \in \Pi$. We say that a function

$$f : F^m \to F^m$$

solves the problem $\Pi$ if and only if

$$\text{graph}(f) \subseteq \Pi.$$

In order to investigate the complexity of a problem we use the model of a computation tree $T$ using the operation symbols $F \cup \{0, 1, +, -, *, /\}$ (multiplications by scalars $\lambda \in F$ included) and the relation symbol $=$ (and $\leq$ when we are working over an ordered field). We define the cost of a computation tree $T$ as the maximum number of multiplications and divisions $T$ performs given an arbitrary input vector. (Compare [12], [15], [16], [18].) The complexity $C(f)$ of a function is then defined as the minimum cost of a tree computing $f$, and finally we put

$$C(\Pi) := \min\{C(f) : f \text{ function solving } \Pi\}$$

for the complexity of the problem $\Pi$. In this paper we focus on the Ostrowski complexity measure which provides enough flexibility to carry through lower bound proofs. However, the upper bounds given in this paper also hold when all operations and comparisons are counted.

One of the leading problems in computational linear algebra is matrix multiplication. In our formal framework

$$MAMU(e, h, l) := \{(A, B, C) \in (F^{e \times h} \times F^{h \times l}) \times F^{e \times l} : AB = C\}$$

Trivially

$$C(MAMU(e, h, l')) \leq C(MAMU(e, h, l)) e' h' l'.$$  

We put

$$M_n := C(MAMU(n, n, n)).$$

(1.1)