AN EFFICIENT ALGORITHM TO RECOGNIZE LOCALLY EQUIVALENT GRAPHS

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To locally complement a simple graph $F$ at one of its vertices $v$ is to replace the subgraph induced by $F$ on $n(v) = \{w : vw \text{ is an edge of } F\}$ by the complementary subgraph. Graphs related by a sequence of local complementations are said to be locally equivalent. We associate a system of equations with unknowns in $GF(2)$ to any pair of graphs $\{F, F'\}$, so that $F$ is locally equivalent to $F'$ if and only if the system has a solution. The equations are either linear and homogenous or bilinear, and we find a solution, if any, in polynomial time.

1. Local equivalence

Let $F$ be a simple graph. The neighborhood of a vertex $v$ of $F$ is $n(v) = \{w : vw \text{ is an edge of } F\}$. To locally complement $F$ at $v$ is to replace the subgraph induced by $F$ on $n(v)$ by the complementary subgraph. We denote by $F* v$ the local complement of $F$ at $v$. Clearly

$$(F * v) * v = F.$$ 

For a word $v_1 v_2 \ldots v_q$ with letters in $V$ we define

$$F * (v_1 v_2 \ldots v_q) = (((F * v_1) * v_2) \ldots) * v_q$$

and we say that $F' = F * (v_1 v_2 \ldots v_q)$ is locally equivalent to $F$. This is actually an equivalence relation because the above equality implies $F = F' * (v_q \ldots v_2 v_1)$. We notice that locally equivalent graphs are defined over the same vertex-set.

Local complementations are natural operations in the following situation. Let $m$ be a word on a set of letters $V$ and suppose that each letter precisely occurs twice in $m$. An alternance of $m$ is a non-ordered pair $xy$ of letters such that we alternatively meet $\ldots x \ldots y \ldots x \ldots y \ldots$ or $\ldots y \ldots x \ldots y \ldots x \ldots$ when reading $m$. The simple graph on the vertex-set $V$ whose edges are the alternances of $m$ is denoted by $A(m)$ and called the alternance graph of $m$. For example if $m = 041213243$ then $A(m)$ has edges 01, 12, 23, 34, 40. Consider some $v \in V$, the decomposition $m = P v Q v R$ where $P, Q, R$ are subwords of $m$, and $m * v = P v Q v R$ where $Q$ is the mirror-image of $Q$. With the preceding example we have $m * 1 = 0412013243$. It is easy to verify that $A(m * v) = A(m) * v$. This interpretation was introduced by A. Kotzig [6], and we give further details in [2]. Not every simple graph is an alternance graph. For example the 5-wheel is not.

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2. Isotropic systems

This section recalls background properties proved in [3], except (2.5) proved in [4].

For any finite set $V$, we consider $\mathcal{P}(V)$, the power-set of $V$, with its canonical structure of vector-space over $GF(2)$. Thus for $X, Y \subseteq V$, $X + Y$ is the symmetric difference of $X$ and $Y$. The neighborhood function of a simple graph $F$ over the vertex-set $V$ is the linear function $n : \mathcal{P}(V) \rightarrow \mathcal{P}(V)$ such that $n(v) = \{w : vw \text{ is an edge of } F\}$, $v \in V$.

Let $K$ denote a 2-dimensional vector space over $GF(2)$, provided with the bilinear form given by $\langle x, y \rangle = 1$ if and only if $0 \neq x \neq y \neq 0$. For any finite set $V$ we consider that the $2^{|V|}$-dimensional vector space $K^V$ is provided with the bilinear form $\langle A, B \rangle = \sum(\langle A(v), B(v) \rangle : v \in V)$. An isotropic system is a pair $S = (L, V)$ where $V$ is a finite set and $L$ is a totally isotropic subspace of $K^V$ (i.e. $\langle A, B \rangle = 0$ for every $A, B \in L$) such that $\dim(L) = |V|$.

A vector $A \in K^V$ is said to be complete if $A(v) \neq 0$ for every $v \in V$. For $P \subseteq V$ let $AP \in K^V$ be defined by $AP(v) = A(v)$ if $v \in P$ and $AP(v) = 0$ in $v \notin P$. Let $\hat{A} = \{AP : P \subseteq V\}$ and notice that $\hat{A}$ is a subspace of $K^V$. If $A$ is complete and $\dim(L \cap \hat{A}) = 0$ then $A$ is called an Eulerian vector of $S$. The reader may refer to [3] for a correspondence between 4-regular graphs and isotropic systems where Eulerian vectors correspond to Euler tours.

Two vectors $A, B \in K^V$ are supplementary if $0 \neq A(v) \neq B(v) \neq 0$ for every $v \in V$. Let $(F, A, B)$ be a triple with a simple graph $F$ and two supplementary vectors $A, B \in K^V$. Where $n$ is the neighborhood function of $F$ and

$L = \{An(P) + BP : P \subseteq V\},$

it is easy to verify that $S = (L, V)$ is an isotropic system (see [3] for details). We call $(F, A, B)$ a graphic presentation of $S$ and $F$ a fundamental graph of $S$.

(2.1) If $(F, A, B)$ is a graphic presentation of an isotropic system $S$, then $A$ is an Eulerian vector of $S$. Conversely if $A$ is an Eulerian vector of $S$, then there exists a graphic presentation $(F', A', B')$ such that $A' = A$, and this graphic presentation is unique.

(2.2) Let $A$ be an Eulerian vector of the isotropic system $S = (L, V)$, and let $v \in V$. There exists precisely one Eulerian vector $A'$ satisfying $A'(v) \neq A(v)$ and $A'(w) = A(w)$ for every $w \in V \setminus \{v\}$.

We use the notation $A * v$ to represent $A'$ of (2.2). For any word $m = v_1v_2 \ldots v_q$ on $V$, we let $A * m = (((A * v_1) * v_2) * \ldots) * v_q$.

(2.3) If $A$ and $A'$ are any two Eulerian vectors of an isotropic system $S = (L, V)$, then there exists a word $m$ on $V$ such that $A' = A * m$.

(2.4) Let $P = (F, A, B)$ be a graphic presentation of an isotropic system $S = (L, V)$, and let $v \in V$. The graphic presentation of $S$ induced by the Eulerian vector $A * v$ is $P * v = (F * v, A + Bv, B + An(v))$ (so that $A * v = A + Bv$).