ASYMPTOTIC ENUMERATION BY DEGREE SEQUENCE OF
GRAPHS WITH DEGREES \( o(n^{1/2}) \)

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Received February 21, 1989

We determine the asymptotic number of labelled graphs with a given degree sequence for
the case where the maximum degree is \( o(E(G)|^{1/3}) \). The previously best enumeration, by the first
author, required maximum degree \( o(E(G)|^{1/4}) \). In particular, if \( k = o(n^{1/2}) \), the number of regular
graphs of degree \( k \) and order \( n \) is asymptotically
\[
\frac{(nk)!}{(nk/2)!2^{nk/2}(k)!^n} \exp\left(-\frac{k^2 - 1}{4} - \frac{k^3}{12n} + O(k^2/n)\right).
\]
Under slightly stronger conditions, we also determine the asymptotic number of unlabelled graphs
with a given degree sequence. The method used is a switching argument recently used by us to
uniformly generate random graphs with given degree sequences.

1. Introduction

Where it is suitable, we will use the notation of [1] or [4]. For any integer
\( y \geq 0 \), define \([x]_y = x(x-1) \cdots (x-y+1)\). Let \( k = k(n) = (k_1, k_2, \ldots, k_n) \) be
a sequence of nonnegative integers with even sum. Define \( k_{\text{max}} = \max_{i=1}^n k_i \) and
\( \bar{k} = (k_1 + k_2 + \cdots + k_n)/n \). For \( r \geq 0 \), further define \( M_r = M_r(k) = \sum_i [k_i]^r \) and
\( \nu_r = \nu_r(k) = \frac{\sum_i k_i^r}{(\bar{k}^r/n)}. \) It is easy to see that \( 1 = \nu_0 = \nu_1 \leq \nu_2 \leq \nu_3 \leq \cdot \cdot \cdot, \) with
the inequalities being equalities if and only if \( k_1 = k_2 = \cdots = k_n \). For simplicity,
write \( M = M_1. \)

Let \( \mathcal{G}(k) \) be the set of all labelled simple graphs with degree sequence \( k \), and
define \( G(k) = |\mathcal{G}(k)|. \) We are concerned with the asymptotic value of \( G(k) \) as
\( n \to \infty. \) Many authors have obtained results by restricting the growth of the
maximum degree. Work prior to [1] can be found summarised there. More recently,
a completely different approach [3] has born fruit for high degrees. Interestingly, the
result in both these extreme cases can be cast in a common form.

**Theorem 1.1.** Define \( \lambda = \bar{k}/(n - 1) \) and \( \gamma_2 = n\bar{k}^2(\nu_2 - 1)/(n - 1)^2. \) Suppose that
either of the following is true:

(i) \( 1 \leq k_{\text{max}} = o(M^{1/4}), \) \( M \to \infty. \)

(ii) \( |k_i - \bar{k}| = O(n^{1/2+\epsilon}) \) and \( \min\{\bar{k}, n - \bar{k} - 1\} > cn/\log n \) for sufficiently small
\( \epsilon > 0 \) and any \( c > 2/3. \)

Then
\[
G(k) \sim \sqrt{2}(\lambda^\lambda(1 - \lambda)^{1-\lambda})^{\left(\frac{\gamma_2}{4}\right)\prod_{i=1}^n \left(\frac{n - 1}{k_i}\right) \exp\left(\frac{1}{4} - \frac{\gamma_2^2}{4\lambda^2(1 - \lambda)^2}\right).}
\]
In this paper we will determine the asymptotic value of $G(k)$ when $k_{\text{max}} = o(M^{1/3})$. The result will match Theorem 1.1 if some extra restrictions are imposed on the amount of variation amongst the degrees. The method used closely resembles that of [1], the major improvement being the use of switching operations which lend themselves to easier analysis. In Section 6, we will consider the case of unlabelled graphs.

2. The model

Consider a set of $M$ points arranged in cells $v_1, v_2, \ldots, v_n$ of size $k_1, k_2, \ldots, k_n$, respectively. Take a partition $P$ (called a pairing) of the $M$ points into $M/2$ parts (called pairs) of size 2. The degree of cell $i$ is $k_i$.

The multigraph $G(P)$ associated with $P$ has vertices $v_1, v_2, \ldots, v_n$. The edges of $G(P)$ are in correspondence with the pairs of $P$; the pair $(x, y)$ corresponds to an edge $(v_i, v_j)$ if $x \in v_i$ and $y \in v_j$. A loop of $P$ is a pair whose two points lie in the same vertex, while a link is one involving two distinct vertices. Two pairs are parallel if they involve the same cells. The multiplicity of a pair is the number of pairs (including itself) parallel to it. A single pair is a pair of multiplicity one. A double pair is a set of two parallel pairs of multiplicity two, whilst a triple pair is a set of three parallel pairs of multiplicity three.

If $p$ is a point, then $v(p)$ is the cell containing that point.

For $l, d, t \geq 0$, define $\mathcal{E}_{l,d,t} = \mathcal{E}_{l,d,t}(k)$ be the set of all pairings with degrees $k$, and exactly $l$ loops, $d$ double pairs, and $t$ triple pairs, but no loops of multiplicity greater than one nor pairs of multiplicity greater than three.

We will make use of the following three operations on a pairing: the first two were introduced in [4].

**I $\ell$-switching:** Take a loop $\{p_1, p'_1\}$ and two links $\{p_2, p'_2\}$ and $\{p_3, p'_3\}$, such that five distinct cells are involved. Replace these three pairs by $\{p_1, p_2\}$, $\{p'_1, p'_3\}$ and $\{p'_2, p_3\}$. It is required that all of the pairs created or destroyed be single. (See Figure 1.)

![Fig. 1. An $\ell$-switching](image)

**II $d$-switching:** Take a double link $\{\{p_1, p'_1\}, \{p_2, p'_2\}\}$, where $v(p_1) = v(p_2)$, and two links $\{p_3, p'_3\}$ and $\{p_4, p'_4\}$, such that six distinct cells are involved. Replace these