COHOMOLOGY OF PROJECTIVE VARIETIES
WITH REGULAR $SL_2$ ACTIONS

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Let $G$ be a complex semisimple linear algebraic group, $B$ a fixed Borel subgroup of $G$, $H$ a maximal torus of $G$ in $B$, $g$ and $h$ the Lie algebras of $G$ and $H$, respectively. Kostant has expressed the cohomology ring of $G/B$ as the coordinate ring $A(N \cap h)$ of the scheme theoretic intersection $N \cap h$ of the variety of nilpotent elements $N$ of $g$ with $h$. The purpose of this paper is to give a similar description of the cohomology ring of a nonsingular complex projective variety $X$ with a "regular" $SL_2$ action. We will show that there is an intrinsically defined subscheme $Z$ of $X$ whose coordinate ring $A(Z)$ is isomorphic to the cohomology ring of $X$. When $X = G/B$, we will identify $A(Z)$ with Kostant's description $A(N \cap h)$.

0. Introduction

One of the most useful aspects of a flag manifold $G/B$ is that its cohomology ring $H^*(G/B, \mathbb{C})$ admits several different descriptions. The classical semi-simple or Borel-Chevalley description says that $H^*(G/B, \mathbb{C})$ is the coinvariant algebra $A(h)/I_W$ associated to the Cartan subalgebra $h$ of $g$. On the other hand, the nilpotent or Kostant description says $H^*(G/B, \mathbb{C})$ is the coordinate ring $A(N \cap h)$ of the scheme theoretic intersection of the nilpotent variety $N \subset g$ with $h$.

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In this paper we will study this semi-simple/nilpotent phenomenon as a special case of what happens when one has an action of $\text{SL}_2(\mathbb{C})$ on a smooth complex projective variety $X$.

We call a holomorphic action of $\text{SL}_2$ on $X$ regular if maximal unipotent subgroups have isolated fixed points. In this case, one knows that any maximal torus also has isolated fixed points and every maximal unipotent has a unique fixed point. We will now describe the general semi-simple/nilpotent situation. Let $B$ denote a Borel subgroup of $\text{SL}_2$ and suppose $V$ and $V_s$ are respectively the holomorphic vector fields generated by the maximal unipotent and maximal torus in $B$. The nilpotent description of $H^*(X, \mathbb{C})$ is given in Proposition 1.1 where it is shown that the coordinate ring $A(Z)$ of the zero scheme $Z$ of $V$ has a canonical grading making it isomorphic in the sense of graded rings with $H^*(X, \mathbb{C})$. In the semi-simple case, however, the coordinate ring $A(Z_s)$ of the variety $Z_s$ of the zeros of $V_s$ is not graded. Rather $A(Z_s)$ admits a filtration $F_0 \subset F_1 \subset \ldots$ such that $F_p F_q \subseteq F_{p+q}$ and

$$\text{Gr } A(Z_s) = \oplus F_p / F_{p-1} \simeq \oplus H^{2p}(X, \mathbb{C}) = H^*(X, \mathbb{C}).$$

For $G/B$, the filtration on $A(Z_s)$ is very well understood. Explicitly, let $h \in g$ be a regular semi-simple element that generates $V_s$. We may assume $h \in k$, so let $W \cdot h$ denote the orbit of $h$ under $W$. The coordinate ring $A(W \cdot h)$ has a natural filtration and a fundamental result is that $A(W \cdot h) \simeq A(Z_s)$ as filtered rings ([3,7]). Thus

$$\text{Gr } A(W \cdot h) \simeq H^*(G/B, \mathbb{C}).$$

It is not hard to see that $\text{Gr } A(W \cdot h)$ is the coinvariant algebra $A(h)/I_W$, so this amounts to the semi-simple description (Proposition 2.2).

In the nilpotent case we may assume that the unique zero of $V$ is given by $B \in G/B$. A natural coordinate system near $B$ is given by $b^{-}_u$ and we may consider the grading on $A(b^{-}_u)$ induced by $V_s$ explained in Proposition 1.1. With respect to this grading, the ideal $I(Z)$ of $Z$ is homogeneous, and we are able to find a graded homomorphism $\phi: A(h) \rightarrow A(b^{-}_u)$. In Theorem 2.2 we show that $\phi$ induces an isomorphism of graded rings $\tilde{\phi}: A(h)/I_W \rightarrow A(b^{-}_u)/I(Z)$.  

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