Research Papers

Some general optimal design results using anisotropic, power law nonlinear elasticity*

P. Pedersen
Department of Solid Mechanics, Technical University of Denmark, DK-2800 Lyngby, Denmark

Abstract Recent results on optimal design with anisotropic materials and optimal design of the materials themselves are in most cases based on the assumption of linear elasticity. We shall extend these results to the nonlinear model classified as powerlaw elasticity. These models return proportionality between elastic strain energy density and elastic stress energy density. This is shown to imply localized sensitivity analysis for the total elastic energy, and for a number of optimal design problems this immediately gives practical, general results.

For two- and three-dimensional problems the effective strain and the effective stress are defined from an energy consistent point of view, and it is shown that a definition generalizing the von Mises stress must be used. The optimization criterion of uniform energy density also holds for nonlinear materials, and several general conclusions can be based on this fact. Applications to size design illustrate this.

For stiffness optimization the ultimate optimal material design problem is addressed. The validity of recent results are extended to nonlinear materials, and a simple proof based on constraint on the Frobenius norm is given. We note that the optimal material is orthotropic, that principal directions of material, strain and stress are aligned, and that there is no shear stiffness. In reality, the constitutive matrix only has one nonzero eigenvalue and the material therefore has stiffness only in relation to the specified strain condition. Results related to orientational design with orthotropic materials are also focused on.

With respect to strength optimization, i.e. the more difficult problem with local constraints, we shall comment on the influence of the different strength criteria.

1 Introduction
The increasing use of anisotropic materials and the ability to tailor material to specific needs have presented a challenge to research on optimal design. Results have been available at least since the early eighties and some of these results are available in textbooks, see Haftka et al. (1990).

In addition to the traditional parameter classification of optimal design as size, shape and topology design, the advanced materials make it necessary to deal with orientational design and with micromechanics design of the material itself.

In most cases, the optimizations with the advanced materials are based on linear elasticity or on perfect plasticity. For early research on the sensitivity analysis related to nonlinear and transient problems, see Cardoso and Arora (1988) or Choi and Santos (1987). For a more recent overview, see Michaleris et al. (1994).

The goal of the present paper is to put forward some general results that can be used without numerical sensitivity analysis. That is, we want to deal with nonlinear problems but with the simplest possible extension from linear elasticity, which is the power-law nonlinear elasticity. Because this class of problems can also be used to describe plasticity theory in the deformation plasticity formulation and stationary creep, this simple extension covers a broad range of practical important problems.

A number of new results are included in the paper. From the localized sensitivity analysis the coalignment of principal directions of strain, stress and material, also follows for nonlinear elasticity. So do the extension of the ultimate optimal material design. For a class of problems without bounds on the design variables the optimal designs are independent of the power of the nonlinearity. We focus on the possible changes in optimal design when the von Mises stress constraint is applied. A more general discussion of the alternative effective stress/strain measures for anisotropic and/or compressive materials is included and the paper ends with comments on shape optimization for minimum energy concentration.

2 Analysis, effective stress/strain and energy densities
The analysis of anisotropic, nonlinear elastic structures/continua will be presented in the secant formulation, as described by Pedersen and Taylor (1993). We shall here concentrate on the compliance matrix and the definition of effective stress. The preferred effective stress is defined in an energy-consistent way and the relation to the von Mises stress is pointed out. Then, the very important relation between strain and stress energy densities is established. This relation is not well-known, but was already mentioned by Hill (1956). We shall use the following notation: for the stress and strain vectors: \( \{ \sigma \} \) and \( \{ \varepsilon \} \), which for 3D-problems, each have 6 elements; for the scalar effective stress, strain the notation \( \sigma_e, \varepsilon_e \); and for the constant reference modulus of elasticity \( E \). The nonlinearity is described by the powers \( n \) or \( p \), where \( n = 1/p \).

2.1 The nondimensional compliance matrix
In the compliance description (see end of this section)
\[
\{ \varepsilon \} = \sigma_e^{n-1} E^{-n} [\beta] \{ \sigma \},
\]

* Dedicated to Professor Franz Ziegler on the occasion of his 60-th birthday in December 1997
\[ \sigma_2^2 := \{\sigma\}^T [\beta]\{\sigma\}, \]  
the nondimensional matrix \([\beta]\) describes the anisotropy, and the only restriction on the matrix is that it must be symmetric and positive semidefinite, i.e.

\[ [\beta]^T = [\beta] \quad \text{and} \quad [\beta] \succeq 0, \]  

Now, separating the stress vector into normal terms with index \(N\) and shear terms with index \(S\), we have

\[ \{\sigma\}^T \begin{bmatrix} \{\sigma\}_N^T \\ \{\sigma\}_S^T \end{bmatrix} = \begin{bmatrix} \{\sigma_{11,22,33}\} \\ \sqrt{2}\{\sigma_{12,13,23}\} \end{bmatrix}, \]  

and, accordingly, we have the following \([\beta]\) matrix:

\[ [\beta] = \begin{bmatrix} [\beta]_{NN} & [\beta]_{NS} \\ [\beta]_{NS} & [\beta]_{SS} \end{bmatrix}, \]  

with the submatrices

\[ [\beta]_{NN} = \begin{bmatrix} \beta_{1111} & \beta_{1122} & \beta_{1133} \\ \beta_{1122} & \beta_{2222} & \beta_{2233} \\ \beta_{1133} & \beta_{2233} & \beta_{3333} \end{bmatrix}, \]  

\[ [\beta]_{NS} = \sqrt{2} \begin{bmatrix} \beta_{1111} & \beta_{1113} & \beta_{1123} \\ \beta_{2212} & \beta_{2213} & \beta_{2223} \\ \beta_{3312} & \beta_{3313} & \beta_{3323} \end{bmatrix}, \]  

\[ [\beta]_{SS} = 2 \begin{bmatrix} \beta_{1112} & \beta_{1113} & \beta_{1123} \\ \beta_{1123} & \beta_{1313} & \beta_{1323} \\ \beta_{1223} & \beta_{2323} & \beta_{3323} \end{bmatrix}. \]  

In the orthotropic directions the orthotropic case is characterized by the simpler form

\[ [\beta]_{SS} = 2 \begin{bmatrix} \beta_{1112} & 0 & 0 \\ 0 & \beta_{1113} & 0 \\ 0 & 0 & \beta_{2323} \end{bmatrix}, \quad \text{[\beta]_{NS} = [0]}, \]  

and is thus described by only 9 parameters. The isotropic case is described by only 2 parameters with

\[ [\beta]_{NN} = \begin{bmatrix} \beta_{1111} & \beta_{1122} & \beta_{1122} \\ \beta_{1122} & \beta_{1111} & \beta_{1122} \\ \beta_{1122} & \beta_{1122} & \beta_{1111} \end{bmatrix}, \]  

\[ [\beta]_{NS} = [0]. \]  

For the following analysis we shall list the conditions of incompressibility of an orthotropic description for any stress state, which is

\[ \begin{align*} 
& \beta_{1111} + \beta_{1122} + \beta_{1133} = 0, \\
& \beta_{1122} + \beta_{2222} + \beta_{2233} = 0, \\
& \beta_{1133} + \beta_{2233} + \beta_{3333} = 0, 
\end{align*} \]  

while incompressibility in relation to hydrostatic pressure is obtained by the single condition

\[ \beta_{1111} + \beta_{2222} + \beta_{3333} + 2(\beta_{1122} + \beta_{1133} + \beta_{2233}) = 0. \]  

2.2 The von Mises effective stress

In traditional plasticity theory the effective stress is not defined as shown in (1), but instead by means of the von Mises stress \(\sigma_2^2\) defined by

\[ \sigma_2^2 = \frac{3}{2} \{s\}^T [\beta]\{s\}, \]  

where \(\{s\}\) is the vector of deviatoric stresses, i.e. the hydrostatic pressure is eliminated. Pedersen (1987) shows that this deviatoric stress vector can be obtained by a projection with the projection matrix \([P]\) \(([P]^T = [P]\) and \([P][P] = [P]\))

\[ \{s\} = [P]\{\sigma\}, \]  

and comparing this to the definition of \(\sigma_2^2\) by (1), we only have

\[ \sigma_2^2 = \sigma_2^2 \]  

for the specific compliance matrix that corresponds to an isotropic and incompressible material,

\[ [\beta]_{\text{isotropic and incompressible}} = \begin{bmatrix} 1 & -0.5 & -0.5 & 0 & 0 & 0 \\ -0.5 & 1 & -0.5 & 0 & 0 & 0 \\ -0.5 & -0.5 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1.5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1.5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1.5 \end{bmatrix}. \]  

For other materials there will generally be a difference between the von Mises stress \(\sigma_2^2\) and our energy-based effective stress \(\sigma_e\),

\[ \sigma_2^2 - \sigma_2^2 = \begin{bmatrix} \beta_{1111} & \beta_{1122} & \beta_{1133} \\ -1 & +0.5 & +0.5 \\ \beta_{1122} & \beta_{2222} & \beta_{2233} \\ +0.5 & -1 & +0.5 \\ \beta_{1133} & \beta_{2233} & \beta_{3333} \\ +0.5 & +0.5 & -1 \end{bmatrix} \{\sigma\}^T \]  

here related to an orthotropic description.

In optimal design the solution will naturally depend on the chosen objective and constraints. Therefore, with \(\sigma_e \neq \sigma_2^2\), we obtain solutions related to the specific choice. We shall return to this and just here point out that the more general results are based on the energy related definition \(\sigma_e\).

2.3 The Hill strength measure

The Hill (1950) strength reference \(F_{\text{Hill}}\) for anisotropic materials is

\[ F_{\text{Hill}} = F(\sigma_{22} - \sigma_{33})^2 + G(\sigma_{33} - \sigma_{11})^2 + \\
H(\sigma_{11} - \sigma_{22})^2 + 2L\sigma_{23}^2 + 2M\sigma_{13}^2 + 2N\sigma_{12}^2, \]  

which, in matrix notation with the definition (3), is written

\[ F_{\text{Hill}} = \]