Essential and Limitational Production Factors

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I. Introduction

In his pathbreaking article "Proof of the Law of Diminishing Returns", Shephard (1970) proves under certain axioms for a one-output many-input production structure that a factor combination is essential (for production) if and only if it is weak limitational. A factor combination, i. e., a proper nonempty subset of inputs, is essential if no application of these factors imply no output and a factor combination is weak limitational if there exists a positive bound on these factors such that output is bounded, when the other factors may vary freely.

Shephard assumes that the efficient subsets of the technology are bounded for each output level. This assumption rules out the CES production functions

\[ u = A \cdot \left[ \delta \cdot x_1^{-\rho} + (1 - \delta) \cdot x_2^{-\rho} \right]^{-1/\rho} \]

\( A > 0, \ 0 < \delta < 1, \) with positive elasticity of substitution equal to or smaller than one, i. e., \( \delta \in [0, +\infty) \). These production functions are frequently applied in economics and it is desirable to find a weaker axiom for which a factor combination is essential if and only if it is weak limitational.

In this paper such a condition (beyond certain axioms on the production structure) is introduced. This condition is weaker than the assumption of bounded efficient subsets and loosely worded it asserts that at no positive output level, can the essential factor

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Shephard (1970) also discusses strong limitationality, a factor combination is strong limitational if each positive bound on these factors bounds output. A characterization of strong limitationality is given and it is shown that for a homothetic production structure, a factor combination is strong limitational if and only if it is weak limitational.

2. The Production Technology

A production technology transforming factors of productions (inputs) \( x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n_+ \) into net output \( u \in \mathbb{R}_+ \) is modelled by a production function \( r : \mathbb{R}^n_+ \rightarrow \mathbb{R}_+ \) or inversely by an input correspondence \( u \rightarrow L(u) \subset \mathbb{R}^n_+ \), where \( r(x) \) is the maximal output obtained from the input vector \( x \) and \( L(u) \) denotes all input vectors yielding output rate \( u \), see Shephard (1967). The production function \( r \) and the input correspondence \( L \) are inversely related by

\[
L(u) = \{x : r(x) \geq u\} \quad \text{and} \quad \phi(x) = \max \{u : x \in L(u)\}.
\]

The production function \( \phi \) is here assumed to satisfy the following subset of axioms stated by Shephard (1974):

\[
\begin{align*}
\phi.1 & \quad \phi(0) = 0, \\
\phi.2 & \quad \phi(x) \text{ is bounded for } ||x|| < +\infty, \\
\phi.3 & \quad \phi(\lambda \cdot x) \geq \phi(x) \text{ for } \lambda \geq 1, \\
\phi.4 (a) & \quad \phi(x) > 0 \text{ for some } x, \\
& \quad \text{(b) if } \phi(x) > 0, \phi(\lambda \cdot x) \rightarrow +\infty \text{ as } \lambda \rightarrow +\infty, \\
\phi.5 & \quad \phi \text{ is upper semi-continuous.}
\end{align*}
\]

In addition to the production function \( \phi \) and the input correspondence \( L \) the following subset of \( L(u) \) is important.

\[
E(u) = \{x \in \mathbb{R}^n_+ : x \in L(u), y \leq x \rightarrow y \notin L(u)\} \quad \text{for } u > 0
\]

The efficient subset

\[
E(0) = 0.
\]

1 Note that from the definition of \( \phi \) it follows that \( L(u) \subset L(v) \) for \( u \geq v \).

2 \( || \cdot || \) denotes the Euclidean norm.

3 \( x \geq y \) denotes \( x \equiv y \) but \( x \neq y \).