SPECTRAL OPERATORS GENERATED BY DAMPED HYPERBOLIC EQUATIONS

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We announce a series of results on the spectral analysis for a class of nonselfadjoint operators, which are the dynamics generators for the systems governed by hyperbolic equations containing dissipative terms. Two such equations are considered: the equation of nonhomogeneous damped string and the 3-dimensional damped wave equation with spatially nonhomogeneous spherically symmetric coefficients. Nonselfadjoint boundary conditions are imposed at the ends of a finite interval or on a sphere centered at the origin respectively. Our main result is the fact the aforementioned operators are spectral in the sense of N. Dunford. The result follows from the fact that the systems of root vectors of the above operators form Riesz bases in the corresponding energy spaces. We also give asymptotics of the spectra and state the Riesz basis property results for the nonselfadjoint operator pencils associated with these operators.

1. Introduction.

As is well known, at the present time there is no general spectral theory of nonselfadjoint operators in a Hilbert space. Such a theory has been constructed only for some special classes of nonselfadjoint operators [1]. One of such classes is the class of spectral operators [2]. While the general abstract theory of such operators has been developed a long time ago, there is still a problem of finding specific examples of the operators of this class. In particular, it turned out that for many natural nonselfadjoint differential operators it is very difficult to verify the spectral property (see, e.g., [3, 4]).

In the present work we formulate our results which provide a class of nontrivial examples of spectral operators. These operators are the dynamics generators for systems governed by hyperbolic equations containing first order damping terms. For these operators we also give an explicit information about asymptotic behavior of the spectrum.
In this work we consider two damped hyperbolic equations. The first of them is the 1-dimensional wave equation, which governs the vibrations of a nonhomogeneous string with nonconstant damping, modulus of elasticity and density coefficients. We consider the equation with a 1-parameter family of boundary conditions which contains, in particular, Dirichlet, Neumann and Sommerfeld radiation conditions. Depending on the value of the parameter, the eigenvalues and the eigenvectors of the corresponding nonselfadjoint operator describe either the complex eigenfrequencies and eigenmodes of a finite string or resonances and resonance states in the scattering of elastic waves on a semi-infinite string. The case of a string with nonconstant damping, constant density and Dirichlet boundary conditions was recently studied in \[10\]. For the case of a nonconstant density, zero damping coefficient and damping in the boundary conditions see \[5-9\]. However, the combination of a non-constant damping and density with nonselfadjoint boundary conditions makes the problem significantly more complicated even if the coefficients are smooth.

The second equation we consider in this paper is the 3-dimensional damped wave equation with spherically symmetric damping and density coefficients and with linear non-selfadjoint boundary conditions on a sphere centered at the origin. Certainly, the spherical symmetry allows separation of variables in spherical coordinates. However, the spectral analysis of the dynamics generator in this case is significantly more complicated than in the aforementioned case of 1-dimensional string equation.

Now we recall the definition of a spectral operator. The definition we give below is a particular case of the general definition. However, it is sufficient for our examples.

**DEFINITION 1**  
a) Let \(\{\psi_n\}_{n=1}^{\infty}\) be a Riesz basis in a complex Hilbert space \(H\). Denote by \(\{\psi_n^*\}_{n=1}^{\infty}\) the unique biorthogonal basis defined by the relations: \(\langle \psi_n, \psi_m^* \rangle = \delta_{nm}\). Let \(\{\lambda_n\}_{n=1}^{\infty}\) be a sequence of complex numbers. Define an operator \(S\) in \(H\) by:

\[
S \varphi = \sum_{n=1}^{\infty} \lambda_n \langle \varphi, \psi_n^* \rangle \psi_n, \quad D(S) = \left\{ \varphi \in H : \sum_{n=1}^{\infty} |\lambda_n|^2 |\langle \varphi, \psi_n^* \rangle|^2 < \infty \right\}.
\]

The operators of the type (1.1) are called scalar operators.

b) An operator \(\mathcal{L}\) in \(H\) is called spectral operator if it can be represented in the form:

\[
\mathcal{L} = S + N
\]

where \(S\) is a scalar operator and \(N\) is a bounded finite rank nilpotent operator (i.e., there exists \(k\) such that \(N^k = 0\)), which commutes with \(S\).