A New Derivation of the Marshalek-Okubo Realization of the Shell-Model Algebra $SO(2v+1)$ for Even and Odd Systems with $v$ Single-Particle Levels

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In recent years, the method for unitarizing nonunitary Dyson boson realizations of shell-model algebras has been both generalized and substantially simplified through the introduction of overtly group-theoretical methods. In this paper, these methods are applied to the boson-odd-particle realization of the algebra $SO(2v+1)$ for $v$ single-particle levels, adapted to the group chain $SO(2v+1) \rightarrow SO(2v) \rightarrow U(v)$, which Marshalek first derived by brute force summation of a Taylor expansion and later Okubo by a largely algebraic technique.

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1. Introduction

The shell-model algebra that describes both even and odd nuclei is that corresponding to the group $SO(2v+1)$ for a space of $v$ single-particle levels. A boson or boson-fermion realization of this algebra requires not only the mapping of pair-transfer and density (multipole) operators, which are bilinear in the fermions, but also the mapping of single-fermion operators. Such a mapping for the bilinear operators only was first given by Dönau and Janssen [1] in a generalized Dyson form. Marshalek [2-4] first obtain the unitary or generalized Holstein-Primakoff (GHP) mappings for both the bilinear and the single fermion operators by summing a Taylor expansion, while Okubo [5] provided a nonperturbative algebraic derivation of both the Dyson and GHP realizations. More recent work of Dobaczewski [6] utilizing coherent state representations recovers results that are equivalent to those of Marshalek and Okubo, but with no effort made to prove the equivalence.

In this note, we present a new derivation of the GHP formulas of Marshalek and Okubo, which we regard as a distinct simplification compared to the earlier derivations. Toward this end, we utilize a new group-theoretical formulation of the mapping problem similar to those that have recently been applied to $SO(2v)$ by Rowe and Carvalho [7] and by Bonatsos, Klein and Zhang [8]. Our approach differs from that of Okubo in the following ways. An irreducible representation (irrep) of a Lie algebra may be characterized completely in one of two ways. One way, that followed by us, is to specify the eigenvalues of the Casimir invariants. The second way, used by Okubo, is to characterize the representation matrices by a sufficient number of independent algebraic relations in addition to the commutation relations. (The most familiar illustration of this difference is the case of spin $1/2$, where the statement $S^2=(3/4)\hbar^2$ contains the same information as the anticommutation relations satisfied by the Pauli spin matrices.) Another important difference is that in Okubo's approach a basis for the physical subspace must first be constructed from a knowledge of the Dyson realization in order to unitarize the latter, while in our approach this step is unnecessary.
The algebra to be studied is generated from \( v \) fermion creation operators \( a_i^\dagger \) \( (i = 1, \ldots, v) \) and the associated annihilation operators \( a_i \) with vacuum \( |0> \).

These operators obey the usual anticommutation rules

\[
\{a_i, a_j\} = \delta_{ij}, \quad \{a_i^\dagger, a_j^\dagger\} = 0. \tag{1.1}
\]

Consequently, the pair operators

\[
A_{ij} = -A_{ji}^\dagger = a_i a_j^\dagger, \quad A_{ij} = -A_{ji}^\dagger = x_j x_i \tag{1.2}
\]

and the density operators

\[
D_{ij} = D_{ji}^\dagger = a_i^\dagger x_j \tag{1.3}
\]

corresponds to the partition

\[
[f_1, f_2, \ldots, f_v] = [1, \ldots, 1, 0, \ldots, 0], \tag{1.12}
\]

with \( N \) ’s and \( v - N \)’s, i.e., the Young pattern with one column of boxes. The totality of vectors (1.10) provides a basis for one of the spinor irreps of \( SO(2v + 1) \) having dimensionality \( 2^v \) [5].

In Sect. 2 we review the generalized Dyson mapping for the algebra studied in this paper, providing some details usually glossed over or replaced by an “ansatz”. Section 3 contains the mathematical preliminaries necessary to simplify the unitarization, which is subsequently carried out in Sect. 4. The results obtained in Sect. 4, however, are not in the form given by Marshalek, but, as shown in Sect. 5, a further application of the results of Sect. 3 yields the necessary transformations. In Sect. 6 we add some concluding observations.

Before commencing with the main task, a word of caution is in order. We shall be mapping from the original fermion space into what is called the ideal boson-odd particle (IBOP) space. Since this mapping is injective, the ideal space contains a physical subspace, which is the faithful image of the fermion space, and an unphysical subspace, which is the orthogonal complement, having nothing to do with fermions. Throughout the derivation, we shall work only within the physical subspace. For this reason, no explicit reference will be made to the unphysical subspace, or will the customary projector to the physical subspace be attached to operators, which will be assumed to be defined on the physical subspace. Another way of stating this is that any operator \( Q \) defined in the physical subspace may be written as \( PQP \) in the whole ideal space, where \( P \) is the projector to the physical subspace. If the operator \( Q \) leaves the physical subspace invariant then \( PQP =QP \), whereas if it leaves the unphysical subspace invariant then \( PQP =PQ \).

Certainly, if \( Q \) is a group generator or if it belongs to the enveloping algebra then it leaves the physical subspace invariant.

2. Review of the Generalized Dyson Mapping

In accord with all the previous work cited, one introduces for each fermion-pair operator \( a_i^\dagger a_j^\dagger \) and its Hermitian conjugate (H.c.) \( x_j x_i \) the elementary boson operators \( b_{ij}^\dagger \) and \( b_{ij} \) in the ideal space, which are antisymmetric in their indices and satisfy the commutation relations

\[
[b_{ij}, b_{kl}] = \delta_{ik} \delta_{jl} - \delta_{il} \delta_{jk},
\]

\[
[b_{ij}, b_{kl}^\dagger] = [b_{ij}^\dagger, b_{ki}] = 0. \tag{1.1}
\]