The Energy Injection into a Fluid by Stochastic Volume Forces and Random Stirring Forces

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The wavenumber spectrum of the stationary energy injection rate into an incompressible fluid described by the Navier-Stokes equations is evaluated for some simple realizations of stochastic volume as well as stirring forces. A general relation between energy injection, fluid's response, and force correlations is derived which was previously shown to be particularly simple for Gaussian distributed forces with white noise frequency spectrum. For two kinds of such model volume forces the energy injection rates are calculated: Fluid volume elements of variable size around randomly chosen positions are forced in one model centrally symmetrically in the other one anti-symmetrically under inversion with various force density profiles. The circumstances under which both models display an energy injection rate \( \sim k^{-1} \) into a band \( dk \) around the wavenumber \( k \) are discussed. As a simple realization of stochastic stirring forces externally moved hard spheres immersed in the fluid are considered. The equation of motion and energy balance for the velocity field of the combined system is discussed. The spectral distribution of energy injection by stirring is shown to be that of a volume force model.

I. Introduction

The velocity field of a fluid described by the Navier-Stokes equations [1] (NSE) decays to zero due to viscous dissipation of momentum and kinetic energy into molecular degrees of freedom unless both are resupplied. The energy- and momentum input into the velocity fluctuations of the fluid flows which are investigated [2-6] in nature and in laboratories to understand turbulence [2-8] stems from the action of surface forces if body forces like thermal buoyancy are absent: Either a finite mean velocity field exerts Reynolds stresses [2, 4] upon the fluctuating components thereby transferring energy to them or surface forces exerted by boundaries and obstacles like grids in a windtunnel do work on the fluid. However, the velocity fluctuations of many of these flows when viewed in the appropriate frame of reference are neither generated continuously nor isotropically but rather represent decaying anisotropic turbulence. Thus, following the work of Kraichnan [9], Wyld [10], and Edwards [11] a statistically defined external random volume force field \( f(r,t) \) has commonly been introduced [12-18] into statistical theories of turbulence in order to provide for a statistically stationary isotropic energy injection.

Such a force field \( f(r,t) \) appears explicitly in the momentum balance of the velocity field. Hence the NSE becomes a nonlinear Langevin equation describing an externally driven stochastic process. The probability distribution of \( f(r,t) \) when chosen to be invariant under time and space translation and under arbitrary rotations enforces a statistically stationary, homogeneous, isotropic, fluctuating velocity field. The latter adjusts itself in a statistically stationary state to the forcing such as to guarantee dissipation of energy at precisely the rate it is put in by the forces. The often heard statement that the random forces supply the energy which is dissipated by viscous stresses is misinterpreted if the former are thought to be adjusted to the latter. The converse is true – e.g. in the case of Gaussian distributed forces with white frequency spectrum the energy input into
any wavenumber band being solely determined by the external forces can be chosen arbitrarily. The velocity field amplitudes react according to the NSE in direct response to the external field and via the action of the nonlinear mode-coupling terms and thus are capable to set up an energy flow in Fourier space by which in every wavenumber band any injection is balanced against transfer and dissipation.

These statistical theories [9-20] of turbulence have to cope with the absence of a physically motivated guideline which restricts the choice of the driving force as, e.g., in stochastic processes simulating fluctuations around thermal equilibrium [21]. In contrast to turbulent velocity fluctuations the former obey detailed balance and a fluctuation dissipation theorem [21, 22]. Hence the statistics of the driving forces is prescribed [22-24] via potential conditions and Einstein relations to ensure a stationary probability distribution of the thermal fluctuations of the form $e^{-\beta H}$. Furthermore, is the physical concept for introducing fluctuating forces much better founded in this case: they represent the microscopic degrees of freedom which have been projected out [25] in the Langevin equation for the macrovariables. The relevance of thermal fluctuations upon turbulence has also been investigated lately [26, 27] but we will not add to the discussion [28] of internally generated force fluctuations in the externally unforced nonlinear NSE. Their energy supply to those wavenumbers relevant in turbulence is negligible in comparison with injection by external random forces since the latter has to sustain a velocity field on a macroscopic scale large enough to entail high Reynolds numbers [2-8].

A physically well founded concept for the external forcing device, however, has not been developed so far. Many authors, including the present one, have made more or less vague allusions to stirring forces but energy injection via external random surface forces has, to the best of our knowledge, not been discussed explicitly. On the other hand is turbulence, generated nonrandomly in cylindrical tanks by rotating impellers, investigated experimentally [29].

Practically all random forcing devices introduced into theories and numerical simulations have been Gaussian distributed volume forces with white frequency spectrum. Their rate of energy input into the velocity field was either chosen, for mathematical convenience, to be power-law distributed over wavenumbers or to display a step function behavior restricting the energy injection to low-wavenumber bands. The latter choice is motivated by the Kolmogorov picture [30-32] of turbulence in which large-scale fluid motion (eddies) decay predominantly into smaller eddies of comparable size thereby transporting energy in a sequence of many eddy decay steps to the very small-scale motion where dissipation takes place. In this picture a band limited forcing is singled out by the following commonly adopted argumentation: It is reasonable to expect the statistical properties of the large-$k$ velocity fluctuations to be insensitive to details of the artificially introduced external forcing provided the latter generates directly only small-wavenumber modes such that large-$k$ modes receive their energy predominantly from those with smaller wavenumbers. However, a unanimously accepted solution to both problems invoked in the above argumentation, the validity of the Kolmogorov picture and the relation between forcing and the statistical dynamics of velocity fluctuations, has not yet been given (c.f. Fig. 1 of [33]) despite their central importance in the statistical theory of fully developed turbulence.

Since the net energy flow in $k$-space is dictated by the distribution of the external energy injection rates over wavenumber bands we feel it is worthwhile to investigate the latter in some detail. So in this work the wavenumber spectrum of the stationary energy injection rate is evaluated for some reasonable model realizations of external random volume as well as surface-forces.

In Sect. II we list basic properties of the fluid's velocity field in the presence of an external random volume force field: The momentum and energy balances and the statistical description of the velocity field are reviewed. The correlation functions which represent energy injection, dissipation, and transfer rates and their role in the balance of energy flow through a wavenumber band are discussed together with their symmetry properties. A relation between energy injection, fluid response, and random force correlation derived in Appendix A for general forced stochastic processes is shown to reduce for Gaussian distributed forces with white noise spectrum to a particularly simple, well known form in which the fluid response drops out. The energy injection rate into volume element $dk$ in Fourier space is then given by the spectrum $D(k)$ of the force field correlation.

In Sect. III $D(k)$ is evaluated for two kinds of model realizations of Gaussian distributed, white noise volume forces: Finite fluid volume elements of various sizes around random positions are forced in one model centrally symmetrically, in the other model antisymmetrically under inversion with different force density field profiles. The force field correlations $D(k)$ decompose naturally into self and coherent correlations. Their contributions to the energy injection are evaluated for various degrees of field co-