Ditkin’s Condition and Primary Ideals in Central Beurling Algebras

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Abstract

In this note some questions of ideal theory for the center and more generally the $B$-fixed subalgebras of a Beurling algebra $L^1_\omega(G)$ are discussed. Sufficient conditions on $\omega$ are given for these subalgebras to satisfy Ditkin’s condition, or for primary ideals to be maximal or at least of finite codimension.

Let $G$ be a locally compact group with Haar measure $dx$, and let $B$ be a group of automorphisms of $G$, containing the group $I(G)$ of inner automorphisms, such that $G$ is an $[FC]$ group; that is, the orbits $B[x]$ are relatively compact for each $x \in G$. Let $\omega$ be a weight function on $G$, that is a measurable, locally bounded function satisfying $\omega(x) \geq 1$ and $\omega(xy) \leq \omega(x)\omega(y)$ for all $x, y \in G$. Then one can define the Beurling algebra $L^1_\omega(G) = \{f \in L^1(G): \|f\|_\omega = \int_G |f(x)|\omega(x)dx < \infty\}$, a subalgebra of $L^1(G)$ under convolution, and a Banach algebra in the norm $\| \cdot \|_\omega$. In previous papers [5, 6] J. Liukkonen and the author have studied the center $Z L^1_\omega(G)$ and the $B$-fixed subalgebra $Z^B L^1_\omega(G) = \{f \in L^1_\omega(G): f \circ \beta = f \text{ in } L^1_\omega \text{ for each } \beta \in B\}$. These algebras are always semisimple commutative Banach algebras, and a sufficient condition on the rate of growth of $\omega$ was given in [5] for $Z^B L^1_\omega(G)$ to be regular and Tauberian. The question of whether Ditkin’s condition is satisfied, or whether closed primary ideals are maximal, was treated only for the case $\omega = 1$ [6, § 3]. An example was given there for which closed primary ideals are not maximal, and a class of groups was given for which Ditkin’s condition holds in $Z L^1_\omega(G)$ (hence, in particular, for which closed, primary ideals are maximal).

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The purpose of this note is to find more general circumstances in which Ditkin's condition, or its consequence concerning primary ideals, holds in $\mathcal{Z}^B L^1_\omega(G)$.

The results in [5] concerning $\mathcal{Z}^B L^1_\omega(G)$ were, for the most part, derived under the assumption that $\omega$ is $B$-invariant. This assumption is somewhat inconvenient, and for many purposes it seems preferable to replace it by a structural assumption on $G$ and $B$. Thus we shall assume, in the remainder of this note, that $G$ is an $[FC]_W$ group, and

that $B$ consists of automorphisms of bounded displacement;

that is, $\{\beta x \cdot x^{-1} : x \in G\}$ is relatively compact for each $\beta \in B$. When $B = I(G)$ itself this assumption is automatically fulfilled so the results of this paper hold for any $[FC]$-group. On the other hand if $G$ is an abelian group without compact subgroups (except the trivial subgroup) then this assumption forces $B$ to consist only of the identity automorphism [3, (3.18)].

Under the hypothesis that $B$ acts by automorphisms of bounded displacement, one can prove that all the results in [5] from (1.3) through (3.1) are still valid without the assumption that the weight function $\omega$ be $B$-invariant. Since we shall want to make use of this assertion later we give a quick sketch of the proof. First one shows that every weight function on $G$ is (under the given hypothesis) at least locally $B$-invariant. More precisely, the weight function $\omega_0 = \sup_{\beta \in B} \omega \circ \beta$ is $B$-invariant, and for each open, compactly generated, $B$-invariant subgroup $H \subset G$, $\omega|_H$ is equivalent to $\omega_0|_H$ (i.e. for some constants $c, d > 0$, $c \omega|_H \leq \omega_0|_H \leq d \omega|_H$). Indeed, since $H$ is a compactly generated $[FC]$-group it contains a maximal compact subgroup $K$; $K$ is characteristic in $H$, and $H/K$ is abelian and aperiodic [3, (3.20)]. Consequently $B$ acts trivially on $H/K$, that is, $B[x] \subset xK$ for each $x \in H$, so $\omega(x) \leq \omega_0(x) \leq (\sup_K \omega) \omega(x)$ on $H$. From this one shows easily that $Z^B C_\omega(G)$ is $\| \cdot \|_\omega$-dense in $A = Z^B L^1_\omega(G)$: for a given $f \in A$, one considers the subgroup generated by a sufficiently large compact subset of $\text{supp} f$, and applies [5, (1.3)]. The next step is to show that the weight function $\Omega(x) = \lim_{n \to \infty} \omega(x^n)^{1/n}$ defined in [5, § 2] is $B$-invariant, and that a continuous $B$-spherical function $\varphi$ on $G$ [5, (1.4)] is bounded by $\omega(\| \varphi \|_\infty < \infty)$ if and only if $\varphi$ is bounded by $\Omega(\| \varphi \Omega \|_\infty < \infty)$. This is a pointwise condition so it also can be checked on appropriately chosen compactly generated subgroups, using [5, (2.2)]. The