Multipliers with Singularities Along a Curve in $\mathbb{R}^n$

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(Received 23 January 1984)

Abstract. We consider in $\mathbb{R}^n, n > 2$, the curve $\gamma(t) = (t, t^2, \ldots, t^n), -\eta_0 \leq t \leq \eta_0,$ $\eta_0 > 0$ a small number. We study the boundedness of operators $T_{\beta}, \beta > 0,$ defined by multipliers which present singularities along $\gamma(t).$ Our results are derived from a sharp estimate on a suitable maximal function. In the case $n = 2$ the $T_{\beta}$'s are Bochner–Riesz operators and our results coincide with the known ones.

Introduction. Bochner–Riesz spherical summation operators $U_{\beta}, \beta > 0,$ are defined in $\mathbb{R}^n$ by the formula $(U_{\beta}f)^{\wedge}(\xi) = u_{\beta}(\xi) \hat{f}(\xi)$ where

$$u_{\beta}(\xi) = \begin{cases} \text{dist}(\xi, \Sigma_{n-1})^\beta, & \text{if } \|\xi\| < 1 \\ 0, & \text{otherwise} \end{cases}$$

and $\Sigma_{n-1}$ is the unit sphere of $\mathbb{R}^n, n \geq 2.$ For $n = 2$ the results on the boundedness of these operators acting on $L^p(\mathbb{R}^2)$ are sharp. The core of the proof [4] is very much similar to the proof of the following restriction theorem of the Fourier transform of an $L^p$ function to the unit circle $\Sigma_1.$ If $1 \leq p < 4/3$ and $1/q > 3(1 - 1/p)$ then

$$\|\hat{f}\|_{L^q(\Sigma)} \leq c_p \|f\|_{L^p(\mathbb{R}^2)}. \quad (1)$$

For the proof of this we refer the reader to [3]. The curvature of $\Sigma_1$ plays an essential role. If we replace $\Sigma_1$ by a $C^\infty$ (compact) curve $\sigma$ with never vanishing curvature then (1) still holds with the constant $c_p = c_p(\sigma).$ Similarly one can define “Bochner–Riesz operators” with respect to $\sigma.$ They satisfy the same estimate of $U_{\beta},$ see [4] and also [7].

In [5] we proved a restriction theorem for the Fourier transform to smooth (compact) curves in $\mathbb{R}^n$ with never vanishing curvatures. In what follows we define the associated multipliers for the curve $\gamma(t) = (t, t^2, \ldots, t^n)$ and we study the boundedness of the corresponding operators acting on $L^p(\mathbb{R}^n).$ Since the first curvature $k_1(t) \sim 2$ for $t \sim 0$ there exists an arc $\Gamma$ of equation $\gamma(t), -\eta_0 \leq t \leq \eta_0$ and a small
cylinder around $\Gamma$, that we denote by $C(\Gamma)$, such that for any $\xi \in C(\Gamma)$ there exists only one $t = t(\xi) \in [-\eta_0, \eta_0]$ with the normal hyperplane at $\vec{v}(t(\xi))$, i.e. the one orthogonal to $\vec{v}'(t(\xi))$, going through $\xi$. Let $(\xi_1', \xi_2', \ldots, \xi_n')$ be the coordinates of $\xi \in \mathbb{R}^n$ in the Frenet frame $\vec{v}_1(t), \vec{v}_2(t), \ldots, \vec{v}_n(t)$ of $\Gamma$. For $\xi \in C(\Gamma)$ we define our multipliers, depending upon a parameter $\delta > 0$, by the formula

$$m_\delta(\xi) = \left\{ \prod_{i=2}^n |\xi_i' (\xi)| \frac{2^n}{\delta/2(n-1)!} G(\xi) \right\}$$

where $G(\xi)$ is a smooth cut off function of value one on the ball centered at the origin of radius $\eta_0/20$ and zero outside the double of it.

Observe that these multipliers are naturally adapted to the geometry of the curve. Namely consider any point $\vec{v}(t)$, $t \in [-\eta_0, \eta_0]$, and a box $B$ centered at $\vec{v}(t)$ of dimensions $2^{-k/2}, 2^{-2k/2}, \ldots, 2^{-k}$ along $\vec{v}_1(t), \vec{v}_2(t), \ldots, \vec{v}_n(t)$ respectively then $|m_\delta(\xi)| \leq c_n 2^{-k/4}$ for $\xi \in B$. We define operators $T_\delta$ by the formula $(T(f))^\wedge(\xi) = m_\delta(\xi) \check{f}(\xi)$ for all $f \in C_0^\infty(\mathbb{R}^n)$. We study the boundedness of $T_\delta$ (Theorem 2) by introducing a suitable maximal function on which we give sharp estimates (Theorem 1). In the case $n = 2$ our result coincides with the classical one on Bochner–Riesz operators and in the case $n = 3$ we obtain the result of [6]. The method used here is similar to that of [6], but we need a more general theorem on the maximal function that we derive from a result of [2].

The result. We start with the maximal function that controls the problem. In [2] the following lemma has been proved:

**Lemma 1.** Let $\vec{v} : [0, 1] \to \Sigma_{n-1}$ be a smooth curve crossing a finite number of times each hyperplane of $\mathbb{R}^n$. Given a number $N \gg 1$ consider the points $\vec{v}_j = \vec{v}(j/N)$, $j = 1, 2, \ldots, N$. For any $d > 0$ consider

$$M_{N,d}f(\vec{x}) = \sup_j (2d)^{-1} \int_{-d}^d |f(\vec{x} + t \vec{v}_j)| dt.$$ 

Then there exists a constant $c$ independent of $N, d, f$ such that for every $f$ in $L^2(\mathbb{R}^n)$ the following inequality holds

$$\| M_{N,d}f \|_2 \leq c \log N \| f \|_2.$$ 

Now let $t_j, j = 1, \ldots, N$ be $N$ values uniformly distributed on $[-\eta_0, \eta_0]$ and let $B_{N,d}$ be the family of boxes $R = R_j$ of dimensions $dN, dN^2, \ldots, dN^n$ along $\vec{v}_1(t), \vec{v}_2(t), \ldots, \vec{v}_n(t)$, $j = 1, \ldots, N$. We define for $d > 0$