CONTRACTIVE HOMOMORPHISMS AND TENSOR PRODUCT NORMS

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For any complex domain \( \Omega \), one can ask if all contractive algebra homomorphisms of \( \mathcal{A}(\Omega) \) (into the algebra of Hilbert space operators) are completely contractive or not. By Ando's Theorem, this has an affirmative answer for \( \Omega = D^2 \), the bi-disc - while the answer is unknown for \( \Omega = (\ell^1(2))^1 \), the unit ball of \( \mathbb{C}^2 \) with \( \ell^1 \) - norm. In this paper, we consider a special class of homomorphisms associated with any bounded complex domain; this well known construct generalizes Parrott's example. Our question has an affirmative answer for homomorphisms in this class with \( \Omega = (\ell^1(2))^1 \). We show that there are many domains in \( \mathbb{C}^2 \) for which the question can be answered in the affirmative by reducing it to that of \( \Omega = D^2 \) or \( (\ell^1(2))^1 \). More generally, the question for an arbitrary \( \Omega \) can often be reduced to the case of the unit ball of an associated finite dimensional Banach space. If we restrict attention to a smaller subclass of homomorphisms the question for a Banach ball becomes equivalent to asking whether in the analogue of Grothendieck's inequality, in this Banach space, restricted to positive operators, the best constant is 1 or not. We show that this is indeed the case for \( \Omega = D^2, D^3 \) or the dual balls, but not for \( D^n \) or its dual for \( n \geq 4 \). Thus we isolate a large class of homomorphisms of \( \mathcal{A}(D^3) \) for which contractive implies completely contractive. This has many amusing relations with injective and projective tensor product norms and with Parrott's example.

1 CONTRACTIVE HOMOMORPHISMS

Let \( \Omega \subseteq \mathbb{C}^m \) be a bounded domain, and \( \mathbb{C}^{n \times n} \) be the \( n \times n \) matrices over the complex field. For \( \omega \) in \( \Omega \) and \( A_1, \ldots, A_m \) in \( \mathbb{C}^{n \times n} \), let

\[
\langle \nabla f(\omega), A \rangle = A_1 \frac{\partial f}{\partial z_1}(\omega) + \cdots + A_m \frac{\partial f}{\partial z_m}(\omega), \quad f \in H^\infty(\Omega),
\]

where \( A = (A_1, \ldots, A_m) \). The map \( \varphi_\omega(A, \cdot) : H^\infty(\Omega) \to \mathbb{C}^{2n \times 2n} \), defined by

\[
\varphi_\omega(A, f) = \begin{pmatrix} f(\omega)I_n & \langle \nabla f(\omega), A \rangle \\ 0 & f(\omega)I_n \end{pmatrix}
\]

is easily seen to be an unital algebra homomorphism, which is continuous from \( H^\infty(\Omega) \) equipped with the topology of uniform convergence on compact sets into \( \mathbb{C}^{2n \times 2n} \) with the
usual operator norm topology. In this paper, we study a stronger notion of continuity. Indeed, we investigate the norm of each of the operators

$$\varphi^{(k)}_\omega(A, \cdot) = \varphi_\omega(A, \cdot) \otimes I : H^\infty(\Omega) \otimes C^{k \times k} \to (C^{2n \times 2n} \otimes C^{k \times k} \cong C^{2nk \times 2nk}, \text{op}),$$

where $\|F\| = \sup\{\|F(z)\|_\text{op} : z \in \Omega\}$, for $F \in H^\infty(\Omega) \otimes C^{k \times k}$. Let

$$\|\varphi_\omega(A, \cdot)\|_{cb} = \lim_{k \to \infty} \|\varphi^{(k)}_\omega(A, \cdot)\|.$$

The map $\varphi_\omega(A, \cdot)$ is said to be contractive if $\|\varphi_\omega(A, \cdot)\| \leq 1$ and is completely contractive if $\|\varphi_\omega(A, \cdot)\|_{cb} \leq 1$.

It is an important open problem to determine domains $\Omega \subseteq C^m$ for which every contractive homomorphism $\varphi_\omega(A, \cdot)$ is completely contractive.

These homomorphisms are related to the familiar notion of a spectral set and complete spectral set for the commuting operator tuple

$$N(\omega, A) = \left( \begin{array}{cc} \omega_1 I_n & A_1 \\ 0 & \omega_1 I_n \end{array} \right), \ldots, \left( \begin{array}{cc} \omega_m I_n & A_m \\ 0 & \omega_m I_n \end{array} \right).$$

Thus, for any rational function $r$ in the algebra $\mathcal{R}(\tilde{\Omega})$ of rational functions with poles off $\tilde{\Omega}$, the evaluation map $r \to r(N(\omega, A))$, $r \in \mathcal{R}(\tilde{\Omega})$ is well defined and coincides with the homomorphism $\varphi_\omega(A, \cdot)$ on $\mathcal{R}(\Omega)$.

Here are two competing definitions of spectral set (resp. complete spectral set). We would say that the operator tuple $N(\omega, A)$ admits the compact set $\Omega$ as a spectral set (resp. complete spectral) set if

1. the homomorphism $\varphi_\omega(A, \cdot)$ is contractive (resp. completely contractive) on the algebra $\mathcal{A}(\tilde{\Omega})$ of functions holomorphic in a neighbourhood of $\tilde{\Omega}$,
   or if

2. the homomorphism $\varphi_\omega(A, \cdot)$ is contractive (resp. completely contractive) on the algebra $\mathcal{R}(\Omega)$.

Agler [1] uses the first definition, while Paulsen [7] uses the second one.

These two notions of a spectral set need not coincide. We will be mainly concerned with the homomorphism $\varphi_\omega(A, \cdot)$ as a map on $H^\infty(\Omega)$. Agler [1] points out that if $\varphi_\omega(A, \cdot)$ is contractive over the algebra $H^\infty(U)$ for every open set $U$ containing $\tilde{\Omega}$, then $\varphi_\omega(A, \cdot)$ is contractive over $\mathcal{A}(\tilde{\Omega})$. However, for certain domains, if the homomorphism $\varphi_\omega(A, \cdot)$ or for