FINITE SECTIONS FOR
SINGULAR INTEGRAL OPERATORS
BY WEIGHTED CHEBYSHEV POLYNOMIALS

Peter Junghanns, Steffen Roch, Uwe Weber

A finite section method for the approximate solution of singular integral equations with
piecewise continuous coefficients on intervals is considered. The problem is transformed
in such a way that results which were previously obtained for singular integral equations
on the unit circle using localization methods in Banach algebras are applicable to it.
Thus, necessary and sufficient conditions for the stability of the approximation method
can be proved.

1 INTRODUCTION

By $\mathbb{R}$ and $\mathbb{C}$ we denote the real and complex number fields, respectively, and $\mathbb{R}^+$ is the half-line
$\{t \in \mathbb{R} : t > 0\}$. The symbol $T$ stands for the unit circle: $T = \{t \in \mathbb{C} : |t| = 1\}$. In the following, $\Gamma$
is one of the curves $T$, $\mathbb{R}$, $\mathbb{R}^+, [-1, 1]$. If $\Gamma = [-1, 1]$, the symbol $\Gamma$ will often be omitted. $L^2(\Gamma)$ and
$L^\infty(\Gamma)$ denote the usual Lebesgue spaces. By $C(\Gamma)$ and $PC(\Gamma)$ we mean the spaces of continuous
and of piecewise continuous functions, respectively. The latter space consists of all functions having
an at most countable number of jumps and being continuous at all other points. If $\Gamma$ is bounded,$PC(\Gamma)$ coincides with the closure in $L^\infty(\Gamma)$ of the set of all functions with a finite number of jumps.

As usual, the singular integral operator of Cauchy type on $\Gamma$ is defined by

$$(S_\Gamma u)(x) := \frac{1}{\pi i} \int_{\Gamma} \frac{u(t)}{t - x} \, dt.$$  

It is well-known that $S_\Gamma \in \mathcal{L}(L^2(\Gamma))$ (see e. g. [3]), where $\mathcal{L}(X)$ denotes the Banach algebra
of all linear bounded operators on a Banach space $X$. Furthermore, we define the operators
$P_\Gamma := \frac{1}{2}(I + S_\Gamma)$, $Q_\Gamma := I - P_\Gamma$, where $I$ denotes the identity operator.
We consider four important and well-known examples of Jacobi weights defined on the open interval \((-1,1)\):

\[
\begin{align*}
\omega_1(x) &= (1 - x^2)^{1/4}, \\
\omega_2(x) &= (1 - x^2)^{-1/4}, \\
\omega_3(x) &= (1 + x)^{1/4}(1 - x)^{-1/4}, \\
\omega_4(x) &= (1 - x)^{1/4}(1 + x)^{-1/4}.
\end{align*}
\]

By \(U_n^{(\nu)}\) we denote the normalized orthogonal polynomial of degree \(n\) with positive leading coefficient with respect to the weight \(\omega_n^2\) on \((-1,1)\) (\(\nu = 1, \ldots, 4\)). These polynomials can be described by

\[
\begin{align*}
U_n^{(1)}(\cos \xi) &= \sqrt{\frac{2}{\pi}} \sin((n + 1)\xi) / \sin \xi, \\
U_n^{(2)}(\cos \xi) &= \sqrt{\frac{2}{\pi}} \cos(n\xi) \quad (n \geq 1), \\
U_n^{(3)}(\cos \xi) &= \sqrt{\frac{\sin(n + 1/2)}{\sin 1/2}}(\cos(n + 1/2)\xi) / \cos \xi, \\
U_n^{(4)}(\cos \xi) &= \sqrt{\frac{\sin(n + 1/2)}{\sin 1/2}}(\sin(n + 1/2)\xi) / \sin \xi.
\end{align*}
\]

(1.1)

Then the functions \(\{\tilde{u}_n^{(\nu)}\}_{n=0}^{\infty}\) defined by

\[
\tilde{u}_n^{(\nu)}(x) = w_n(x)U_n^{(\nu)}(x)
\]

form a complete orthonormal system in \(L^2\) for \(\nu = 1, \ldots, 4\). Finally, we introduce a sequence of Fourier projections \(P_n^{(\nu)}\) by

\[
P_n^{(\nu)} \sum_{k=0}^{\infty} \xi_k \tilde{u}_k^{(\nu)} := \sum_{k=0}^{n-1} \xi_k \tilde{u}_k^{(\nu)}.
\]

Obviously, this sequence converges strongly to \(I\). In the following, we will usually omit the upper index \(\nu\) and use one notation to summarize all the four cases.

We consider equations of the form \(Au = f\) in \(L^2\) with

\[
(Au)(x) = a(x)u(x) + \frac{b(x)}{\pi i} \int_{-1}^{1} \frac{u(t)}{t - x} \, dt,
\]

or

\[
(Au)(x) = a(x)u(x) + \frac{1}{\pi i} \int_{-1}^{1} \frac{b(t)u(t)}{t - x} \, dt,
\]

(1.2)

(1.3)

where \(u \in L^2\) is the unknown function and \(f \in L^2\) and \(a, b \in PC\) are given. The conditions on the coefficients \(a\) and \(b\) that guarantee the invertibility of the operators (1.2), (1.3) are well-known (see [3, Chapt. 9, Theorem 4.1]). In the following, the operators (1.2) and (1.3) will briefly be referred to as \(aI + bS\) and \(aI + SbI\), respectively.

We replace the equation \(Au = f\) by the discrete approximate equations

\[
A_n u_n := P_n AP_n u_n = P_n f
\]

(1.4)

in the finite-dimensional subspaces \(im P_n\). What we ask about is the convergence of this so-called finite section method, i.e. we ask if for every \(f \in L^2\) the equations (1.4) have a unique solution \(u_n\).