On Sums of Two Coprime $k$-th Powers

By

Werner Georg Nowak, Wien

(Received 12 January 1989)

Abstract. Under the assumption of the Riemann Hypothesis, an asymptotic formula with a sharp error term is established for $\sum_{n \leq x} \varphi_k(n)$, where $\varphi_k(n)$ denotes the number of ways to write $n$ as a sum of two $k$-th powers of coprime positive integers ($k \geq 3$).

1. Introduction

For a fixed natural number $k \geq 2$, let $r_k(n)$ denote the number of integer pairs $(u, v)$ with $|u|^k + |v|^k = n$ and $\varphi_k(n)$ the number of $(u, v) \in \mathbb{N}^2$ with $\gcd(u, v) = 1$ and $u^k + v^k = n$. Concerning the mean values of these arithmetic functions, it is known that

$$
\sum_{1 \leq n \leq x} r_k(n) = A_k x^{2/k} + B_k x^{1/k - 1/k^2} \sum_{n=1}^{\infty} n^{-1-1/k} \sin \left(2 \pi n x^{1/k} - \frac{\pi}{2k} \right) + O(x^{25/38k + \varepsilon})
$$

for every $\varepsilon > 0$, where

$$A_k = \frac{2 \Gamma^2(1/k)}{k \Gamma(2/k)} \quad \text{and} \quad B_k = 2^{3-1/k} \pi^{-1-1/k} k^{1/k} \Gamma \left(1 + \frac{1}{k} \right).$$

(See [11] and cf. the literature cited there for earlier results with only marginally worse error terms. For $k = 2$, the second term is of course meaningless and the error term can be sharpened to $O(x^{7/22 + \varepsilon})$, according to MOZZOCHI and IWANIEC [9].) Since, by the Moebius inversion formula,

$$\varphi_k(n) = \frac{1}{2} \sum_{m_1, m_2 | n} \mu(m_1) r_k(m_2)$$

(2)
for \( n \geq 2 \), it can be deduced by elementary means from the deep estimate
\[
\sum_{m \leq y} \mu(m) = O(y \exp(-c(\log y)^{3/5} (\log \log y)^{-1/5})), \quad c > 0,
\] (3)
(cf. Walfisz [16], p. 191) that
\[
R_k(x) = \sum_{1 \leq n \leq x} \varrho_k(n) =
\]
\[
= \frac{3}{2\pi^2} A_k x^{2/k} + O(x^{1/k} \exp(-c'(\log x)^{3/5} (\log \log x)^{-1/5})).
\] (4)
(See Krätzel [6], Theorem 1.) At our present state of knowledge about the zeros of the Riemann zeta-function, it is not possible to improve the exponent \( \frac{1}{k} \) of \( x \) in this error term: If \( Z(s) \) denotes the generating function of \( r_k(n) \), then (by (2)) \( \frac{Z(s)}{4\pi^2 (ks)} \) generates \( \varrho_k(n) \) (for \( n \geq 2 \)), and this could have simple poles arbitrarily close to the line \( \Re s = \frac{1}{k} \). (Cf. the argument by Evelyn and Linfoot [1].)

Thus it may be of interest to ask to which extent (4) can be improved if we assume the Riemann Hypothesis to be true. Replacing (3) by the classic conditional estimate
\[
\sum_{m \leq y} \mu(m) = O(y^{1/2+\varepsilon}), \quad \varepsilon > 0,
\] (5)
one can prove, by the same elementary convolution argument, that
\[
R_k(x) = \frac{3}{2\pi^2} A_k x^{2/k} + O(x^{1/k-1/k(3k+2)+\varepsilon})
\] (6)
for \( k \geq 3 \). This was noticed by Krätzel [6], as a special case of a theorem due to Moroz [8].

For \( k = 2 \), one can apply complex integration and an idea of Montgomery and Vaughan [7]. This depends on the fact that \( Z(s) = \sum r_2(n) n^{-s} \) is Epstein's zeta-function, and was carried out in [14] to establish an asymptotic for \( R_2(x) \) with the order term \( O(x^{15/38+\varepsilon}) \).