THE SPECTRAL FACTORIZATION PROBLEM FOR MULTIVARIABLE DISTRIBUTED PARAMETER SYSTEMS

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This paper studies the solution of the spectral factorization problem for multivariable distributed parameter systems with an impulse response having an infinite number of delayed impulses. A coercivity criterion for the existence of an invertible spectral factor is given for the cases that the delays are a) arbitrary (not necessarily commensurate) and b) equally spaced (commensurate); for the latter case the criterion is applied to a system consisting of two parallel transmission lines without distortion. In all cases, it is essentially shown that, under the given criterion, the spectral density matrix has a spectral factor whenever this is true for its singular atomic part, i.e. its series of delayed impulses (with almost periodic symbol). Finally, a small-gain type sufficient condition is studied for the existence of spectral factors with arbitrary delays. The latter condition is meaningful from the system theoretic point of view, since it guarantees feedback stability robustness with respect to small delays in the feedback loop. Moreover its proof contains constructive elements.

1 Introduction

In the control literature, much attention has been paid to the spectral factorization problem for linear time invariant lumped - and distributed parameter systems. This problem is met under different forms in several applications. A classical one is the solution of Wiener-Hopf type problems, i.e. systems of integral equations on a half line, with kernels of a specific type. A lot of results in this field have been developed by Gohberg and Krein: see e.g.[18], [22] and the book [17]. Another context where the factorization problem arises is the theory of feedback control system design: LQ-theory and robust feedback stability, see below; another issue may be the multiplier technique in passivity theory (circle criterion, etc...): see e.g. [36], [16]. The specific spectral factorization problem of this paper is motivated by applications in feedback control system design, more precisely in the analysis of closed-loop stability robustness, i.e. the analysis of the graph distance between two possibly unstable systems for obtaining robustness estimates of feedback stability, and in the solution to the Linear-Quadratic optimal control problem by frequency domain techniques for distributed parameter, i.e. infinite-dimensional state-space, systems with bounded or unbounded control.
This paper studies the multivariable spectral factorization problem in the framework of the Calieter-Desoer algebra of possibly unstable distributed parameter system transfer functions (see e.g. the survey paper [11] or the book [15]), whose corresponding subalgebra of proper stable transfer functions is denoted by $\mathcal{A}_\infty$. The starting points are references [8] and [9] where one investigates respectively singlevariable general spectral factorization and multivariable spectral factorization with no delayed impulses. The results obtained in those papers are extended here to multivariable distributed parameter systems with an impulse response having an infinite number of delayed impulses. Criteria for the existence of an invertible spectral factor are given for the cases that the delays are a) arbitrary and b) equally spaced (commensurate case). The analysis of case a) is based on a result of [26] ([1],[2]), while that of case b) is a corollary (already present in [39]). In both cases the condition is the coercivity on the extended imaginary axis of the matrix spectral density. An essential step in the sufficiency proof of this condition is to show that, once the singular atomic part of the spectral density (i.e. its series of delayed impulses) has an invertible spectral factor, then so does the overall spectral density. Indeed, the problem is then reduced to spectral factorization with no delayed impulses, whose solution is known, see [9]. It is also recalled that the coercivity condition implies the existence of invertible matrix spectral factors with entries in a larger algebra than $\mathcal{A}_\infty$, viz. $H_\infty$, see e.g. [25], [34] and references therein. Finally a stronger small-gain type condition is proved to be sufficient for the existence of matrix spectral factors with entries in $\mathcal{A}_\infty$. This last condition is seen to be meaningful from the system theoretic point of view, since it guarantees feedback stability robustness with respect to small delays in the feedback loop. The results are illustrated by some examples. In particular, the results of the case of commensurate delays are applied to a system consisting of two transmission lines in parallel without distortion.

The paper is organized as follows. Section 1 contains the present introduction and a list of notations and abbreviations. The solution of the general spectral factorization problem of spectral densities with arbitrary delays is described in Section 2, whereas the detailed proofs are given in Section 3. The next section is devoted to two particular cases which are important in applications. The final section contains some conclusions.

A list of notations and abbreviations and a remark on the causal-anticausal decomposition of a distribution with support on the real axis are given below.

**List of notations and abbreviations:**

- $\mathbb{R}$, (respectively $\mathbb{R}_-$, $\mathbb{R}_+$) := set of real (respectively nonpositive-, nonnegative-real) numbers;
- $\mathbb{C}$ := field of complex numbers;
- $\mathcal{C}_\sigma$, (respectively $\mathcal{C}_{\sigma+}$) := \{s $\in \mathbb{C} : \text{Re}(s) \geq \sigma$, (respectively $> \sigma$)\} ($\sigma$ is omitted if $\sigma = 0$);
- $\mathcal{S}_\sigma$, (respectively $\mathcal{S}_{\sigma+}$) := \{s $\in \mathbb{C} : \sigma \leq \text{Re}(s) \leq -\sigma$, (respectively $\sigma < \text{Re}(s) < -\sigma$)\};