Extremal Solutions for Nonlinear Parabolic Problems with Discontinuities

By

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Abstract. This paper examines nonlinear parabolic initial-boundary value problems with a discontinuous forcing term, which is locally of bounded variation. Assuming that there exist an upper solution \( \varphi \) and a lower solution \( \psi \), we prove the existence of a maximal and of a minimal solution within the order interval \([\psi, \varphi] \subseteq L^p(T \times I)\). Our approach is based on a Jordan-type decomposition for the discontinuous forcing term and on a fixed point theorem for nondecreasing maps in ordered Banach spaces.

1. Introduction

In [17] Stuart studied semilinear elliptic problems involving discontinuous nonlinearities. It is well-known that such problems need not have a solution even under restrictive hypotheses. Section 4 of Stuart [17] contains results illustrating this. It is then a good idea to replace the original equation by a multivalued version of it. In particular, Stuart isolated a broad class of nonlinearities which led to multivalued versions of the problem which were obtained by filling in only the downward jumps of the original discontinuous forcing term \( f(\cdot) \). So if all the jumps are upward (i.e. \( f(r^+) \geq f(r^-) \) for every \( r \in \mathbb{R} \)), then the original single-valued and the multivalued versions produce the same set of solutions. In his main existence theorem (see Theorem 3.1 in [17]), Stuart established the existence of a maximal and a minimal solution, located in the order interval determined by an upper and a lower solution. In [18], Stuart and Toland developed a variational method for such discontinuous problems based on the nonconvex duality theory of Toland [20]. We should also mention the relevant important works of Rauch [16] and Chang [4], who also deal with semilinear elliptic systems involving discontinuities. Rauch used mollification techniques to establish the existence of a solution located between an upper and a lower solution for a problem in which the discontinuous nonlinearity is not monotone and we only assume that it ultimately increases (i.e. \( \lim_{t \to -\infty} f(t) \leq \lim_{t \to +\infty} f(t) \)). Chang used critical point theory for nondifferentiable functions to study similar problems.
While the stationary (elliptic) problem has been studied rather extensively, the study of the corresponding dynamic (parabolic) problem is lagging behind. Only recently some particular semi-linear cases were considered by Feireisl and Norbury [8] and Carl and Heikkila [3]. In [8] the authors proved uniqueness, non uniqueness and comparison theorems, while in [3] the authors establish the existence of a maximal and of a minimal solution ("extremal solutions") between an upper and a lower solution, using a monotone iterative technique. Compared with the work of Carl and Heikkila [3], our work is a twofold improvement. First, we treat fully nonlinear problems (i.e. the partial differential operator is nonlinear). Secondly, and this probably is the main improvement of our work, our hypotheses on the nonlinearity permit discontinuities in both directions, i.e. the discontinuous right-hand side may jump in both directions. In [3] it is required that for some \( M > 0 \) the map \( x \to f(x) + Mx \) is nondecreasing. This condition is the one that permits the authors to device a generalized (transfinite) iteration method. We should also mention the earlier work of Deijl and Hess [6], who used the method of upper and lower solutions to solve a nonlinear parabolic initial-boundary value problem. Extensions of the work of Deijl and Hess were obtained by Mokrane [13] and Boccardo, Murat and Puel [2]. However none of these works deals with the existence of extremal solutions.

The purpose of this paper is to extend to fully nonlinear parabolic problems the existence theorem of Stuart [17] (see Theorem 3.1 in that paper).

2. Mathematical Preliminaries

Let \( T = [0, b] \) and let \( Z \subseteq \mathbb{R}^N \) be a bounded domain with a \( C^1 \)-boundary \( \Gamma \). In what follows \( D_k = \frac{\partial}{\partial z_k} \) \( k \in \{1, 2, \ldots, N\} \) and \( D = \text{grad} = (D_k)_{k=1}^N \). We consider the following nonlinear parabolic initial-boundary value problem:

\[
\begin{aligned}
\frac{\partial x}{\partial t} - \sum_{k=1}^N D_k a_k(t, z, Dx) &= f(x(t, z)) \quad \text{in } T \times Z \\
x(0, z) &= x_0(z) \quad \text{a.e. on } Z, x|_{T \times \Gamma} = 0.
\end{aligned}
\]

Here \( f : \mathbb{R} \to \mathbb{R} \) is a discontinuous nonlinear perturbation. Our hypotheses on the data of (1) are the following:

- \( H(a) \): \( a_k : T \times Z \times \mathbb{R}^N \to \mathbb{R}, k \in \{1, 2, \ldots, N\} \), are functions such that
  - (i) \( (t, z) \to a_k(t, z, \xi) \) is measurable;
  - (ii) \( \xi \to a_k(t, z, \xi) \) is continuous;
  - (iii) \( |a_k(t, z, \xi)| \leq \beta_1(t, z) + c_1 \|\xi\|^{p-1} \) a.e. on \( Z \) for every \( \xi \in \mathbb{R}^N \) and with \( \beta_1 \in L^q(T \times Z), c_1 > 0, 2 \leq p < \infty, \frac{1}{p} + \frac{1}{q} = 1; \)
  - (iv) \( \sum_{k=1}^N (a_k(t, z, \xi) - a_k(t, z, \xi'))(\xi_k - \xi_k') > 0 \) a.e. on \( T \times Z \) for every \( \xi, \xi' \in \mathbb{R}^N, \xi \neq \xi' \);
  - (v) \( \sum_{k=1}^N a_k(t, z, \xi_k) \geq c_2 \|\xi\|^p - \beta_2(t, z) \) a.e. on \( T \times Z \) with \( c_2 > 0 \) and \( \beta_2 \in L^1(T \times Z) \).

- \( H(f) \): \( f : \mathbb{R} \to \mathbb{R} \) is a function of bounded variation on every compact interval in \( \mathbb{R} \) and \( f(r) \in \hat{f}(r) \) for every \( r \in \mathbb{R} \), where \( \hat{f}(r) = \text{conv}\{f(r^+), f(r^-)\} \) with \( f(r^+) = \lim_{\varepsilon \to 0} f(r + \varepsilon) \) and \( f(r^-) = \lim_{\varepsilon \to 0} f(r - \varepsilon) \).