Convergence Rates in the Law of Large Numbers for Random Variables on Partially Ordered Sets

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Abstract

Let \((A, \leq)\) be a partially ordered set, \(\{X_\alpha\}\) a collection of i. i. d. random variables, indexed by \(A\). Let \(S_\alpha = \sum_{\beta \preceq \alpha} X_\beta\), \(|\alpha| = \text{card } \{\beta \in A, \beta \preceq \alpha\}\). We study the convergence rates of \(S_\alpha / |\alpha|\). We derive for a large class of partially ordered sets theorems, like the following one: For suitable \(r, t\) with \(1/2 < r/t < 1\): \(E |X|^t M(|X|^{r/t}) < \infty\) and \(E X = \mu\) if and only if

\[
\sum |\alpha|^{-r} P (\frac{|S_\alpha| - |\alpha| \mu}{|\alpha|^{r/t}} < \epsilon) < \infty
\]

for all \(\epsilon > 0\), where \(M(x) = \sum \alpha \leq \alpha d(\alpha)\) with \(d(\alpha) = \text{card } \{\alpha \in A, |\alpha| = \alpha\}\).

Introduction

Let \((A, \leq)\) be a denumerable infinite, partially ordered set with i) \(\{\beta \in A, \beta \preceq \alpha\}\) is finite for each \(\alpha \in A\) and ii) \(\{\alpha \in A, |\alpha| = j\}\) is finite for each \(j \in \mathbb{N}\), where \(|\alpha| = \text{card } \{\beta \in A, \beta \preceq \alpha\}\). We define \(d(j) = \text{card } \{\alpha \in A, |\alpha| = j\}\) for \(j \in \mathbb{N}\) and \(M(x) = \sum \alpha \leq \alpha d(\alpha)\) for \(x \geq 1\) and \(M(x) = 1\) for \(0 \leq x < 1\).

Let \(\{X_\alpha, \alpha \in A\}\) be a collection of independent, identically distributed variables with common distribution function \(F(x)\) and with mean 0, if it exists. Define \(S_\alpha = \sum_{\beta \preceq \alpha} X_\beta\). Smythe [4] has studied the a.s. convergence of \(S_\alpha / |\alpha|\) when \(|\alpha| \to \infty\). His first result depends on the existence of \(E M(|X|)\) and some additional conditions on the structure of \(A\). Smythe derived then a sufficient condition for a much larger class of sets \(A\), a theorem analogous to one of Hsu and Robbins and Ender for the linearly ordered case: with the assumption i) and ii) on \(A\) and with a regular varying \(M(x)\), i.e. \(M(x) = x^q L(x)\), where \(L(x)\) is slowly varying at infinity and clearly \(q \geq 1\), then \(E |X| M(|X|) < \infty\) implies \(\sum \alpha P(|S_\alpha| > \epsilon |\alpha|) < \infty\) for all \(\epsilon > 0\). The converse is true, if \(L(x)\) is mono-
tonically increasing. In this paper we will extend this last theorem, analogous to theorems of SPITZER [5], KATZ and BAUM [1, 3]. We assume that \( L(x) \) is monotonically increasing, to get equivalent statements. But we need this assumption only in the proof of one direction.

**Theorems**

**Lemma 1:** For any random variable \( X \) \( E M(|X|^t) < \infty \) is equivalent to \( \sum_x P(|X| > |\alpha|^{1/t}) < \infty \).

**Proof:** It follows from Lemma 2.1 of SMYTHE [4].

Let \( \{X_n\} \) be independent and identically distributed as \( X \) and \( A \) satisfying i) and ii). Let \( M(x) = x^\alpha L(x) \), where \( \alpha \geq 1 \) and \( L(x) \) varies slowly and is monotonically increasing.

**Theorem 1:** If \( 0 < t \leq \min(1, 2/\alpha) \), then the following two statements are equivalent:

a) \( E M(|X|^t) < \infty \),

b) \( \sum_x |\alpha|^{-1} P(|S_\alpha| > |\alpha|^{1/t} \varepsilon) < \infty \) for all \( \varepsilon > 0 \).

If \( 1 \leq t \leq 2/\alpha \), then the following two statements are equivalent:

c) \( E M(|X|^t) < \infty \) and \( EX = 0 \),

d) \( \sum_x |\alpha|^{-1} P(|S_\alpha| > |\alpha|^{1/t} \varepsilon) < \infty \) for all \( \varepsilon > 0 \).

**Proof:** We use the methods of BAUM and KATZ [1]. First we show a \( \rightarrow b \) and c \( \rightarrow d \) and assume without loss of generality that \( \varepsilon = 1 \). Define for \( \beta \leq \alpha \)

\[
X^*_\beta = \begin{cases} 
X_\beta & \text{if } |X_\beta| < |\alpha|^{1/t} \\
0 & \text{else}
\end{cases}
\]

Then

\[
E_\alpha X = E X^*_\beta \quad \text{and} \quad Y_{\alpha \alpha} = X^*_\beta - E_\alpha X.
\]

Then

\[
\sum |\alpha|^{-1} P(|S_\alpha| > |\alpha|^{1/t}) \leq \sum |\alpha|^{-1} P(\exists \beta \geq \alpha: |X_\beta| > |\alpha|^{1/t}) + \sum |\alpha|^{-1} P(|\sum Y_{\alpha \alpha}| > |\alpha|^{1/t} (1 - |\alpha|^{-1/t} E_\alpha X)).
\]

We show that every sum of the right side of the inequality (1) is finite.

\[
\sum |\alpha|^{-1} P(\exists \beta \geq \alpha: |X_\beta| > |\alpha|^{1/t}) \leq \sum P(|X| > |\alpha|^{1/t}) < \infty
\]

from Lemma 1. Analogous to BAUM and KATZ [1] we can show that

\[
|\alpha|^{1 - 1/t} E_\alpha X \xrightarrow{|\alpha| \to \infty} 0 \quad \text{for every } t > 0.
\]