EXACT VACUUM SOLUTIONS OF THE DE WITT EQUATION FOR THE CLOSED
AND OPEN FRIEDMANN MODELS. OPERATOR ORDERING AND THE
SINGULARITY PROBLEM

V. N. Mel'nikov and G. D. Pevtsov

The Wyler–De Witt vacuum equation of quantum cosmology is investigated. Exact
solutions are obtained in the closed and open models. It is demonstrated that the
operator ordering of De Witt results in nonsingular general solutions in both
cases. In the closed model the normalized solution is localized on the Planck
scale and can be used as a model of the preinflationary universe. It is also
demonstrated that the asymptotics of the solution with matter have the same form
as the vacuum solutions. In a definite example of an operator ordering differing
from that of the De Witt a singular solution is obtained.

Recently there has been increased interest in the problems and methods of quantum cos-
{}mology in connection with attempts to solve the old problems of the singular state [2–4]
and the origin of the universe [5, 6], and to explain the rapid expansion of the universe
[7] with new ideas, etc.

The first quantization of a closed isotropic model with a reasonable model of matter in
the form of a conformal scalar field (for small times it is equivalent to an ultrarelativistic
gas and at large times equivalent to dust) was realized in [2], where on the basis of a
particular solution it was shown that the probability of a singular state was exactly zero.
The cosmological constant was first taken into account in quantum cosmology in [8], and it
was discovered that for a positive value (positive vacuum energy density) a potential barrier
arises. Thus the possibility arose of explaining the appearance of the universe in terms of
a tunneling through this barrier.

Now various schemes are widely discussed for the birth of the universe from "nothing,
or more precisely, from this or that model of the vacuum. Stanyukovich proposed the first
model of this type [5], where the origin from the vacuum results from fluctuations of a
planckIon, a closed world with Planck dimensions. Later analogous ideas were put forward and
worked out by a series of authors ([8], Trion in 1973, [6], Zel'dovich in 1982, Vilenkin in
1983, etc.). The possibility that the gravitational interaction itself (the Einstein
Lagrangian) appeared as the result of vacuum effects was shown and considered in [6]. The
mechanism involved the spontaneous breaking of the conformal symmetry of the massless Higgs
field, and, further, the beginning of the evolution from Planck scales was the result of
the spontaneous breaking of the gauge symmetry of the scalar Higgs field.

Here we will examine in detail the exact vacuum solutions [9, 10] of the Wyler–De Witt
equation of quantum cosmology. In a closed model it has the form

\[- \frac{3}{64\pi^2} \frac{\partial^2 \Phi}{\partial x^2} + 12\pi^2 x^{2/3} \Phi = 0, \]

where \( x = R^{3/2}, \Phi(x) = (\frac{2}{3})^{1/2} R^{-1/4} \Psi(R), \) \( R \) is the scale factor, which in the quantum case is a
random quantity, and \( \Psi(R) \) is the sought-after state vector of the universe.

We note that in the general case (for any metric) the secondary constraints are formulated
for the density of the superhamiltonian \( H \), which is the basic part of the Hamiltonian density
of the gravitational field. [1] The use of the total Hamiltonian \( H = J (\pi a_0 + aH) d^3x, \) where
\( -\alpha^2 = g^\omega \), and \( \pi \) is the density of the conjugate momentum to the coordinate, is only possible in the case of a finite volume.

Equation (1) is identical to the following:

\[
\frac{\partial^2 \Phi}{\partial x^2} - 256\pi^4 x^{2\alpha} \Phi = 0.
\]

This equation reduces to the Riccati equation and has the general solution [9, 10]

\[ \Phi(x) = V x Z_{3\alpha}(i 12\pi^2 x^{4/3}) \]

and \( \Psi(R) \) correspondingly has the form

\[ \Psi(R) = RZ_{3\alpha}(i 12\pi^2 R^2) = C_1 R J_{3\alpha}(12\pi^2 R^2) + C_2 R K_{3\alpha}(12\pi^2 R^2), \]

where \( J_{3\alpha} \) is the Bessel function of the first kind with an imaginary argument, \( K_{3\alpha} \) is the MacDonald function. The function \( \Psi(R) \) has the following asymptotic form in the limit \( R \to 0 \):

\[ \Psi(R) \sim R^{1/4} \left( K_{3\alpha}(12\pi^2 R^2) \sim R^{-3/4} \right), \]

i.e., \( \Psi(0) = 0 \). Hence we obtain a nonsingular solution in the most general case. However, the general solution for \( \Psi(R) \) contains the function \( J_{3\alpha}(12\pi^2 R^2) \), which has the following asymptotic limit at infinity

\[ J_{3\alpha}(z) \sim \left( \frac{1}{2\pi z} \right)^{1/4} e^z, \]

i.e., is exponentially increasing. In order to eliminate this solution we must set \( C_2 = 0 \).

Thus the solution for \( \Psi(R) \) with physical content has the form

\[ \Psi(R) = c R K_{3\alpha}(12\pi^2 R^2), \quad \text{where} \quad c = \text{const}. \]

The MacDonald function has the asymptotic limit \( K_{3\alpha}(z) \sim (\pi/2z)^{1/4} e^{-z} \) at infinity. Therefore, the state vector can be normalized to one and the constant \( c \) is determined by the normalization condition

\[ \int_0^\infty |\Psi(R)|^2 dR = \frac{c^2 \Gamma(9/8) \Gamma(3/8) \Gamma(3/4)^2}{48 \sqrt{6} \pi^3 \pi} = 1. \]

We have for the normalized wave function

\[ \Psi(R) = \left[ \frac{48 \sqrt{6} \pi^3 \pi}{\Gamma(9/8) \Gamma(3/8) \Gamma(3/4)} \right]^{1/2} R K_{3\alpha}(12\pi^2 R^2). \]

We obtain the following value for the radius \( R \) of the universe

\[ < R > = \int_0^\infty R |\Psi(R)|^2 dR = \frac{\sqrt{6} \pi}{16 \sqrt{2 + \sqrt{2} \Gamma(9/8) \Gamma(3/8) \Gamma(3/4)^2}} \approx 0.06, \]

and thus the curvature of our space without matter has a nonzero finite average. This is a very important property of the quantum treatment and does not have any classical correspondence, in which the Einstein equations for a closed Friedmann world become meaningless if the matter density is zero. [11]