THE INFLUENCE OF LARGE COALITIONS

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This paper contains two results on influence in collective decision games. The first part deals with general perfect information coin-flipping games as defined in [3]. Baton passing (see [8]), an $n$-player game from this class is shown to have the following property: If $S$ is a coalition of size at most $\frac{n}{3\log n}$, then the influence of $S$ on the game is only $O\left(\frac{|S|}{n}\right)$. This complements a result from [3] that for every $k$ there is a coalition of size $k$ with influence $\Omega\left(\frac{k}{n}\right)$. Thus the best possible bounds on influences of coalitions of size up to this threshold are known, and there need not be coalitions up to this size whose influence asymptotically exceeds their fraction of the population. This result may be expected to play a role in resolving the most outstanding problem in this area: Does every $n$-player perfect information coin flipping game have a coalition of $o(n)$ players with influence $1-o(1)$? (Recently Alon and Naor [1] gave a negative answer to this question.) In a recent paper Kahn, Kalai and Linial [7] showed that for every $n$-variable boolean function of expectation bounded away from zero and one, there is a set of $\Omega(\frac{n\omega(n)}{\log n})$ variables whose influence is $1-o(1)$, where $\omega(n)$ is any function tending to infinity with $n$. They raised the analogous question where $1-o(1)$ is replaced by any positive constant and speculated that a constant influence may be always achievable by significantly smaller sets of variables. This problem is almost completely solved in the second part of this article where we establish the existence of boolean functions where only sets of at least $\Omega\left(\frac{n}{\log^2 n}\right)$ variables can have influence bounded away from zero.

1. Introduction

This paper makes two contributions to the area of influences on collective decision procedures (see [3]). The most intriguing open problem in this area is whether in every $n$-player perfect information coin-flipping game (definitions will be given below) there is a coalition of $o(n)$ players whose influence on the game is $1-o(1)$. In other words, is it true that in every such game there is a negligible minority which, by deviating from random behavior can almost surely dominate the game. Consider for integers $n \geq k$ the largest number $\phi = \phi(n,k)$ such that in every $n$-player coin flipping game there is a coalition of $k$ players with influence $\phi(n,k)$. A theorem of Ben-Or and Linial [3] asserts that $\phi(n,k) > c\frac{k}{n}$ for some constant

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In light of this remark we are essentially dealing with the question: Are there games in which the influence of every coalition is proportional to its fraction of the population of players? And if such games do not exist, at which coalition size does a disproportionate influence necessarily show up?

Now [3] introduced a game ("iterated majority of 3") the analysis of which shows that for \( k = O(n^\alpha) \) where \( \alpha = \log_3 2 \approx 0.63 \ldots \) indeed \( \phi(k, n) = \Theta(k/n) \). In the present paper it is shown, that the same holds for larger \( k \) and in fact \( \phi(k, n) = \Theta(k/n) \) for all \( k = O\left(\frac{n}{\log n}\right) \). For all that is known, this might be the largest \( k = k(n) \) for which this statement holds. The proof is based on a detailed analysis of the so-called baton passing game (Saks [8]) whose description follows: Player \( P_1 \) is the first to hold the baton; a player receiving the baton should pass it at random to a player who had not held it yet. The last player to hold the baton is to flip a coin and this bit is the outcome of the game. Saks showed that influences in this game depend in a strong way on the relationship between the coalition size and \( \frac{n}{\log n} \). In particular, coalitions of size \( o\left(\frac{n}{\log n}\right) \) have influence \( o(1) \), while those of size asymptotically bigger than \( \frac{n}{\log n} \) have influence \( 1 - o(1) \). In this paper the analysis is refined to show that if \( k < \frac{n}{3\log n} \) then the influence of a size \( k \) coalition is only \( O(k/n) \). The proof depends on a (fairly complicated) closed form formula for the probability of a coalition of a given size to win the game. An asymptotic analysis of this formula is then carried out to conclude that no disproportionate influence comes up as long as \( k < \frac{n}{3\log n} \).

If indeed the answer to the above problem is positive, and small dominant coalitions always exist, this clearly resembles a result of Kahn, Kalai and Linial [7]. In that paper it is shown that in simple coin flpping games, or what is the same, in any boolean function a small dominant coalition (set of variables) must exist. Specifically, for any \( n \)-variable boolean function whose expected value is bounded away from zero and one there is a set of \( \frac{n\omega(n)}{\log n} \) variables whose influence is \( 1 - o(1) \), where \( \omega(n) \) can be any function which tends to infinity with \( n \). There is, however, a significant difference between the two situations, of simple and general games. In a simple game there always exists a player with influence \( \Omega\left(\frac{\log n}{n}\right) \) that is asymptotically more than its \( \frac{1}{n} \) fraction of the population of the players (this is the main theorem of [7]). More generally for \( k = O\left(\frac{n}{\log n}\right) \), \( k \)-coalitions exist with influence \( \Omega\left(\frac{k}{n}\right) \). In other words, disproportionate influence is present already for the smaller coalition size. Now if indeed \( \phi(k, n) = 1 - o(1) \) for some \( k = k(n) = o(n) \) (possibly \( \frac{n}{\log n} \)) then for general games a disproportion between a coalition’s size and its influence occurs only for larger cardinalities, and rather suddenly (in terms of \( k \)). The explanation of this phenomenon may well differ from the one which was discovered in simple games.

If on the other hand, the answer to the question is negative, and perfect information games can be constructed with no small dominant coalitions, then it is reasonable to expect the baton passing game to be useful in such constructions,