ON A PROBLEM OF ERDŐS AND LOVÁSZ:
RANDOM LINES IN A PROJECTIVE PLANE*

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Let \( n(k) \) be the least size of an intersecting family of \( k \)-sets with cover number \( k \), and let \( \mathcal{P}_k \) denote any projective plane of order \( k-1 \).

**Theorem.** There is a constant \( A \) such that if \( \mathcal{H} \) is a random set of \( m \geq Ak \log k \) lines from \( \mathcal{P}_k \) then
\[
\Pr (\tau (\mathcal{H}) < k) \to 0 \quad (k \to \infty).
\]

**Corollary.** If there exists a \( \mathcal{P}_k \) then \( n(k) = O(k \log k) \).

These statements were conjectured by P. Erdős and L. Lovász in 1973.

0. Introduction

An old problem of Erdős and Lovász [2] asks, given a positive integer \( k \), what (roughly) is the least \( n = n(k) \) for which there exists an \( n \)-member intersecting family of \( k \)-sets whose cover number is \( k \)? (Recall that a family \( \mathcal{H} \) is intersecting if its members are pairwise nondisjoint; its cover number, \( \tau (\mathcal{H}) \), is the least size of a set meeting all sets of \( \mathcal{H} \). For more on this and many related questions see the excellent survey [3].)

The function \( n(k) \) was introduced in [2], where it was shown that
\[
(0.1) \quad n(k) \geq 8k/3 - 3,
\]
and
\[
(0.2) \quad n(k) \leq 4k^{3/2} \log k \quad \text{if} \quad k - 1 \quad \text{is the order of a projective plane}.
\]

Let \( \mathcal{P}_k \) denote any projective plane of order \( k-1 \) (i.e. having \( k \) points on a line). The upper bound (0.2) is an immediate consequence of

**Theorem 0.** ([2]). If \( \mathcal{H} \) is a random set of \( m \geq 4k^{3/2} \log k \) lines from \( \mathcal{P}_k \) then with high probability \( \tau (\mathcal{H}) = k \).

(That is: \( \mathcal{H} \) is chosen uniformly at random from \( m \)-subsets of the line set of \( \mathcal{P} \); with high probability means with probability tending to 1 as \( k \to \infty \); throughout this paper \( \log \) denotes natural logarithm.)

Here we prove, as conjectured (with \( C \) in place of \( 22 \)) in [2],

**Theorem 1.** If \( \mathcal{H} \) is a random set of \( m \geq 22k \log k \) lines of \( \mathcal{P}_k \) then with high probability \( \tau (\mathcal{H}) = k \).

**Corollary.** \( n(k) = O(k \log k) \) provided there exists a \( \mathcal{P}_k \).

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Of course \( \tau(\mathcal{H}) \leq k \) for any set \( \mathcal{H} \) of lines of \( \mathcal{P} = \mathcal{P}_k \), and equality at least requires that

\[(0.3) \text{ each point of } \mathcal{P} \text{ is on at least two lines of } \mathcal{H}.\]

It is probably true that if one randomly chooses lines \( \ell_1, \ell_2, \ldots \) from \( \mathcal{P} \) then with high probability \( \tau(\{\ell_1, \ldots, \ell_t\}) = k \) as soon as \( \mathcal{H} := \{\ell_1, \ldots, \ell_t\} \) satisfies (0.3) (this happens when \( t \) is about \( 3k \log k \)), but we do not see how to prove this.

As for lower bounds, remarkably, nothing is known beyond (0.1). (As mentioned in [2] and again in [1], even \( n(k) > 3k \) does not seem easy.) Erdős (e.g. [1]) currently offers \$500 for a proof or disproof of \( n(k) = O(k) \).

Curiously, if \( n(k) = O(k) \) then the best examples must be quite different from those considered here. Recent results of the author [4] imply that, if we add to the conditions “intersecting family of \( k \)-sets with cover number \( k \)” the requirement that edge intersection sizes in \( \mathcal{H} \) be bounded by some \( o(k) \), then indeed \( |\mathcal{H}|/k \rightarrow \infty \). In particular, \( |\mathcal{H}| = \Omega(k^{3/2}) \) — probably improvable to \( \Omega(k \log k / \log \log k) \) — whenever \( \mathcal{H} \) is a subset of the line set of \( \mathcal{P}_k \) with \( \tau(\mathcal{H}) = k \).

**Proof of Theorem 1**

A few conventions. We write \( S \) and \( L \) for the point and line sets of \( \mathcal{P} = \mathcal{P}_k \). For \( X, Y \subseteq S \), \( L(X) \) denotes the set of lines meeting \( X \), \( \overline{L}(X) \) the complementary set, and \( L(X,Y) \) the set of lines meeting both \( X \) and \( Y \). For \( x \in S \) we shorten \( L(\{x\}) \) to \( L(x) \), etc., except that when \( x \in X \), we use \( L(x,X) := L(\{x,X \setminus \{x\}\}) \). We set \( Q = |S| = |L| = k^2 - k + 1 \).

In what follows, \( A, B, C, D, E > 0 \) and \( \delta \in (0,1) \) are constants whose values will be set later. We must show that for suitable \( A \) (eventually about 22), if \( \mathcal{H} \) is a random subset of \( L \) of size \( m \geq Ak \log k \), then

\[(1.1) \text{ with high probability } \tau(\mathcal{H}) = k.\]

We assume throughout that \( k \) is large enough to support our assertions.

As in [2], we use the “counting sieve”, proving the somewhat stronger

\[(1.2) \sum \left\{ \Pr(\mathcal{H} \subseteq L(X)) : X \in \binom{S}{k-1} \right\} = o(1).\]

As mentioned in [2], \( X \)'s for which \( \overline{L}(X) \) is very small are easily handled:

\[(1.3) \sum \left\{ \Pr(\mathcal{H} \subseteq L(X)) : X \in \binom{S}{k-1}, |\overline{L}(X)| < k^{3/2} - k \right\} = o(1).\]

**Proof.** (Sketch.) As observed in [2] (see Lemma on p. 625), \( |\overline{L}(X)| < k^{3/2} - k \) implies there is some \( \ell \in L \) such that \( |\ell \cap X| < k^{1/2} \). Noting that \( |\ell \cap X| = t \) implies \( |\overline{L}(X)| \geq t(k-t) \), we find that the left-hand side of (1.3) is less than

\[
\sum_{1 \leq t < \sqrt{k}} Q \binom{k}{t} \binom{Q}{t-1} \left(1 - \frac{t(k-t)}{Q}\right)^{Ak \log k},
\]

which is \( o(1) \) if \( A > 3 \).