We investigate nontrivial \((m, n, k, \lambda)\)-relative difference sets fixed by the inverse. Examples and necessary conditions on the existence of relative difference sets of this type are studied.

1. Introduction

Let \(G\) be a finite group of order \(mn\) with a subgroup \(N\) of order \(n\). A \(k\)-element subset \(D\) of \(G\) is called an \((m, n, k, \lambda)\)-relative difference set (or, in short, an \((m, n, k, \lambda)\)-RDS) in \(G\) relative to \(N\) if the expressions \(d_1d_2^{-1}\), for \(d_1\) and \(d_2\) in \(D\) with \(d_1 \neq d_2\), represent each element in \(G \setminus N\) exactly \(\lambda\) times and represent no nonidentity element in \(N\). The concept of relative difference sets was introduced by Butson [3], [4] and Elliott and Butson [6] as a generalization of difference sets. (For detailed descriptions of difference sets, please consult [2], [8].)

For a subset \(S\) of \(G\), we define \(S(t) = \{gt | g \in S\}\). In this paper, we are going to study a particular type of relative difference sets which are fixed by the inverse, i.e. \(D^{-1} = D\).

In the following, we assume that \(G\) is an abelian group of order \(mn\) and \(N\) is a subgroup in \(G\) of order \(n\) such that \(m > 1\) and \(n > 1\). For any subset \(S\) of \(G\), \(S\) is called reversible if \(S^{-1} = S\); and \(S\) is called nontrivial if \(|S| \neq 0, 1, mn - 1\) and \(mn\).

Firstly, let us summarize the known results on reversible relative difference sets (see [1, Section 6]).

**Proposition 1.1.** Let \(D\) be a nontrivial reversible \((m, n, k, \lambda)\)-RDS in \(G\) relative to \(N\). Then

\(1\) \(m = k = \lambda n\) is a square and \(\lambda\) is even;
\(2\) \(|D \cap gN| = 1\) for all \(g \in G\);
\(3\) \(D(t) = D\) for all \(t\) relatively prime to the order of \(G\);
\(4\) if \(H\) is a proper subgroup in \(N\) of order \(s\) and \(\varphi : G \to G/N\) is a natural epimorphism, then \(\varphi D\) is a nontrivial reversible \((\lambda n, n/s, \lambda n, \lambda s)\)-RDS in \(G/H\) relative to \(N/H\).

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By Proposition 1.1(ii), we learn that $|D \cap N| = 1$, say, $D \cap N = \{h\}$ for some $h \in N$. Since $D^{(-1)} = D$, we have $h^2 = 1$. Hence $D' = hD$ is a reversible relative difference set with $D' \cap N = \{1\}$. Let $E_1 = N \setminus \{1\}$, $E_2 = D' \setminus \{1\}$ and $E_3 = G \setminus (N \cup D')$. It is easy to verify that $\{1, E_1, E_2, E_3\}$ spans a Schur ring over $G$ and hence can be used to construct an association scheme of class 3 (see [10]). This is one of the reasons why we are interested in reversible relative difference sets.

In Section 2, we shall see some examples of reversible relative difference sets. In Section 3, we shall show that there are some severe restrictions on the existence of nontrivial reversible relative difference sets. In particular, $N$ must be a direct factor of $G$ and $n$, the order of $N$, is a power of 2. Furthermore, if $n = 2$, every nontrivial reversible relative difference set is constructed by using a $(4u^2, 2u^2 \pm u, u^2 \pm u)$-difference set where $m = 4u^2$. Finally, in Section 4, some open problems are posed.

2. Examples

There are two known families of reversible relative difference sets.

**Example 2.1.** (Arasu, Jungnickel and Pott [1, Corollary 2.11]) If there exists a reversible $(4u^2, 2u^2 \pm u, u^2 \pm u)$-difference set $C$ in a group $K$. Then

$$D = (\{0\} \times C) \cup (\{1\} \times (G \setminus C))$$

is a reversible $(4u^2, 2, 4u^2, 2u^2)$-RDS in $G = \mathbb{Z}_2 \times K$ relative to $\mathbb{Z}_2 \times \{1\}$.

With the constructions of reversible $(4u^2, 2u^2 \pm u, u^2 \pm u^2)$-difference sets given by Menon [12], Turyn [14], Dillon [5] and Leung and Ma [9], we have a large family of reversible $(4u^2, 2, 4u^2, 2u^2)$-RDS for $u = 2^s3^t$ where $s, t$ are nonnegative integers.

For the second example, we need a finite local ring. Let $R$ be a finite local ring of characteristic 2 with its maximal ideal $I$ generated by a prime element $\pi$. Suppose $R/I \cong \mathbb{F}_{2^d}$, a finite field of $2^d$ elements, and $s$ is the smallest positive integer such that $I^s = (\pi^s) = (0)$.

**Example 2.2.** (Leung and Ma [9, Corollary 4.2]) Let $\{A_1, A_2, \ldots, A_{2^t}\}$ be a partition of $R$ such that, for any coset $a + I^{s-1}$ in $R$ and $i = 1, 2, \ldots, 2^t$, we have $|A_i \cap a + I^{s-1}| = 2^{d-t}$ where $t$ is a positive integer less than or equal to $d$. Define $\varphi : R \rightarrow R$ be a mapping such that $\varphi(\pi^ru) = \pi^r u^{-1}$ for any $r = 0, 1, \ldots, s-1$ and unit $u$ in $R$. Let $H = \{g_1, g_2, \ldots, g_{2^t}\}$ be an elementary 2-group with $2^t$ elements. Then

$$D = \bigcup_{i=1}^{2^t} \{(g_i) \times \{(a, b) \in R \times R \mid \varphi(a)b \in A_i\}\}$$

is a reversible $(2^{2sd}, 2^t, 2^{2sd}, 2^{2sd-t})$-RDS in $H \times R \times R$ relative to $H \times \{0\} \times \{0\}$.*

Example 2.2 gives us a family of reversible relative difference sets with $n = 2^t$ for all $t$.

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* For the case $s = 1$, we have to modify the construction: Use $R \cong F_{2^d}$, replace $I^{s-1}$ by $R$ and define $\varphi(a) = a^{-1}$ for all unit $a$ and $\varphi(0) = 0$. 