The Potts Model on Bethe Lattices

II. Special Topics *

F. di Liberto 1, G. Monroy 1, and F. Peruggi 2
Dipartimento di Fisica dell'Università di Napoli, Napoli, Italy

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We analyze the properties of the q-state ferromagnetic Potts model for real q. The nature of the phase transition at the critical point is first-order for q > 2, and second-order for q = 2. The random-bond percolation limit q → 1, and its second-order-like transition, are not related to the previous behaviour since they arise from non-stable phases of the system. It is suggested that this property characterizes the model on high-dimensional lattices, too.

1. Introduction

The Potts model has recently attracted considerable attention for its intricate complexities and the connections with other systems of physical interest. A summary of results and references can be found in a review article by Wu [1]. In a previous paper [2] (which we refer to as paper I from now on) we have given a solution of the q-state Potts model on Bethe lattices, showing that the relative results are related to the behaviour of the model on high-dimensional lattices. In paper I our analysis was done for integer q ≥ 2 and considering both ferromagnetic and antiferromagnetic interactions, while here we are concerned with the critical properties of the ferromagnetic model when q is treated like a real non-negative parameter.

We first study the nature of the transition at the critical point as a function of q. This is done by considering the discontinuities of the spontaneous magnetization and the specific heat, and the peak of the susceptibility, when q changes. We find that at the critical temperature the system is characterized by first-order transitions for every q ≥ 1, except in the Ising case q = 2 where the transition becomes second-order. On the other hand, on d-dimensional lattices it is expected [3–5] that the transition at the critical point is first-order for q > q_c(d), and second-order for q ≤ q_c(d), the latter result being usually supported by the nature of the q = 2 Ising transition and the q → 1 percolative transition. Since it is known that q_c(d) = 2 for d ≥ 4, the properties of the ferromagnetic Potts model on such real lattices seem strictly connected to those on the (infinite-dimensional) Bethe lattices, except for the disagreement about the order of the transition in the range 1 ≤ q < 2, which in turn is related to the q → 1 behaviour of the model.

To investigate about this discrepancy, we have calculated explicitly the q → 1 limits of the thermal functions of the system. Since the random-bond percolation model on Bethe lattices is also solved directly [6], we gave an explicit check of the well-known proofs of the mapping \{Ferromagnetic Potts model\} \rightarrow \{Random-bond percolation\} [7–10]. It turns out that this mapping has a peculiar feature on Bethe lattices, namely that the percolative limit arises from non-stable phases of the system. Thus the critical point of the model (which is related to the stable phases) is characterized by first-order transitions for 1 ≤ q < 2, although the percolative transition for q → 1 is second-order-like. Since it is possible that the percolative limit arises from non-stable phases on real lattices too (see Sect. 4), one concludes that the actual behaviour of the Potts model on d ≥ 4 lattices may be similar to the behaviour observed on Bethe lattices instead of that commonly accepted.

The method which we use for solving the model is the same introduced in paper I, and reduces the description of the whole system to a single parameter

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1 INFN, sezione di Napoli

2 CISM AND GNSM, Unità di Napoli
renormalization. This point deserves a brief discussion, because the rigorous theory of q-state spin models on Bethe lattices, which has been recently established [11, 12], implies that renormalization of q−1 parameters is necessary and sufficient for the complete description of a system. We have shown elsewhere [13] that the general theory, applied to the Potts model, recovers the single parameter approach as a special case (namely when all the q−1 renormalization parameters are equal). Of course the multi-parameter method allows to detect other fixed points, besides those of paper I, which sometimes minimize the free energy and therefore represent the “true” stable phases of the system. The consequent modification in the phase diagrams are discussed extensively in Ref. 13: for the present purposes the following points are actually significant. (i) In the multi-parameter approach q is necessarily integer, so this method is not suitable for extension to real q, while the single parameter approach is. (ii) For ferromagnetic interactions and integer q ≥ 2, the fixed points found at zero external field H in the single-parameter approach characterize the same phases as those found in the H→0+ limit of the multi-parameter approach*. As a consequence of (i) and (ii), the single parameter approach allows us to obtain correct results and to extend them to real q.

The outline of the paper is as follows. A brief review of the procedure used for the solution of the model, and its extension to real q, are given in Sect. 2. The nature of the transition at the critical point is discussed in Sect. 3, while the random-bond percolation limit q→1 is treated in detail in Sect. 4.

2. The Thermal Functions of the Model

The q-state ferromagnetic Potts model is characterized by the Hamiltonian:

$$-\beta \mathcal{H} = K \sum_{\langle i,j \rangle} \delta_{v_i v_j} + H \sum_i \delta_{v_i 1},$$

(1)

where β is the Boltzmann factor, the first (second) sum runs over bonds (sites) of a lattice, Kronecker delta-functions which relate the spin variables v_i = 1, 2, ..., q are used, and K > 0. The procedure introduced in paper I for the solution of this model on a Bethe lattice (i.e. an infinite connected tree whose sites have the same coordination number σ+1) will be used here, too. We do not give again the details, but limit our description to a brief summary of definitions an results.

Let p_r be the probability that v_i = r, and p_s be the conditional probability that for two adjacent sites one has v_j = s given that v_i = r. All the previous probabilities can be expressed in terms of

$$p_1 = 1/[1 + (q - 1) \varphi b/a], \quad p_{11} = e^\varphi /a,$$

$$p_{21} = 1/b, \quad p_{22} = e^\varphi /b,$$

$$a = e^\varphi + (q - 1) \varphi, \quad b = 1 + (e^\varphi + q^{-2}) \varphi.$$  

(2)

For any choice of K and H, the allowed (≡ real positive) values of the physical parameter φ are obtained by solving the equation:

$$\varphi = e^{-H(b/a)^a}$$

(3)

(there are at most three solutions, each corresponding to a phase of the system). All the thermal functions of the system can be expressed in terms of the above mentioned probabilities. The reader should refer to paper I for the explicit expression of the free energy βΦ, the internal energy βU, the entropy Φ/k (k is the Boltzmann constant), and the pair correlation function g_{ij}(r, s) between two sites i and j, separated by l bonds, in the states v_i = r and v_j = s (see also Ref. 12). Of course the magnetization is

$$M = \frac{\delta (\beta \Phi)}{\delta H} = p_1.$$

(4)

Finally, we also consider the isothermal susceptibility and the specific heat at constant field:

$$\chi = \frac{\delta M}{\delta H} = p_1 (1 - p_1) \frac{1 + (p_{11} - p_{22})}{1 - \sigma (p_{11} - p_{22})},$$

$$\mathcal{C}/k = \frac{\delta U}{\delta (1/\beta)} = \beta U - \beta \frac{\delta (\beta U)}{\delta \beta}$$

$$= \frac{1}{2} K^2 [p_1 (p_{11} + (1 - p_i) p_{22}) - \{1 - [p_1 p_{11} + (1 - p_i) p_{22}] \} + \{H^2 [1 + (p_{11} - p_{22})] - 2(\sigma + 1) HK (p_{11} - p_{22}) + (\sigma + 1) \sigma K^2 (p_{11} - p_{22})^2 \} [p_1 (1 - p_1)/[1 - \sigma (p_{11} - p_{22})]].$$

(5)

One obtains the same expressions for χ and Φ/k through calculation of the magnetization, and energy per site, fluctuations by means of the pair correlation function.

Extension of the above-mentioned functions to real q is trivial, provided we use (2) to write the thermal functions directly in terms of K, H, σ, q, and φ. To avoid ambiguities in the interpretation of results, we give priority to the free energy, i.e. we will always attribute to this function the usual physical meaning.

* Such a limit selects, among the possible ordered phases of the system, the ones where the symmetry breakdown concerns the spin state coupled to the field.