Limit cycles of the ballast resistor caused by intrinsic instabilities

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Spatio-temporal patterns of the ballast resistor are investigated. It is well known that in a voltage-controlled ballast resistor an electrothermal instability leads to stable stationary states consisting of hot and cold domains. Such states may become oscillatory unstable, giving rise to the bifurcation of limit cycles. These limit cycles are not caused by the external circuit but by a recently proposed novel intrinsic mechanism. There are two types of oscillatory instabilities: bulk instabilities and boundary-induced instabilities. The bulk instabilities are caused by resistivities which are not monotonically increasing functions of the temperature. The boundary-induced instabilities occur in small systems with Neumann boundary conditions. To find the bulk instability, experiments with materials showing a metal-semiconductor transition or high-temperature superconductors are suggested. To understand these new phenomena, the equation of motion is reduced to ordinary differential equations where the instabilities can be discussed analytically.

The equation of motion for the temperature profile $T(x, t)$ along the wire of a voltage-controlled ballast resistor is [4]

\[ c_v \partial_t T = \kappa \partial_x^2 T - a(T) + j^2 \rho(T), \]

with

\[ j = U \int_0^L \rho(T(x)) \, dx, \]

where $c_v$ is the specific heat, $\kappa$ the heat conductivity, $a(T)$ the heat loss into the coolant, $\rho(T)$ the resistivity of the wire, $j$ the current density which is assumed to be uniform, $U$ the external voltage (which is the control parameter), and $L$ the length of the wire. For simplicity, we neglect the Thomson effect and the temperature dependence of $c_v$ and $\kappa$. We also neglect inhomogeneities.

In a current-controlled ballast resistor (i.e. $j$ is the control parameter) nonuniform temperature profiles are usually only transient phenomena, whereas in a voltage-controlled ballast resistor stable stationary nonuniform states (usually one hot domain surrounded by two cold domains) exist. Because the ballast resistor has a nonlinear current-voltage characteristic, it is easy to provoke self-sustained oscillations (i.e. limit cycles) by connecting the ballast resistor to a suitably chosen circuit. A current-controlled ballast resistor shunted with an inductance and a resistivity is a well-known example for such circuit-induced oscillations (see [8] and references therein).

In a recent paper [9] I have pointed out the possibility of oscillations in a voltage-controlled ballast resistor caused by an intrinsic (i.e. not circuit-induced) instability. This is a bulk instability because it does not depend on the boundary conditions as long as the widths of the hot and the cold domains are large compared to the thermal length (i.e. $\sqrt[\kappa/\alpha}$, where $\alpha$ is the heat transfer coefficient to the coolant). For such an instability to occur, the resistivity of the wire must be a decreasing function of the temperature at the temperature values of either the hot or the cold domain. For example, assume a superconducting material with a negative slope of $\rho$ in the normal phase. Now increase the temperature of the hot domain
slightly. This leads to a decrease of the resistance. Therefore the current increases. If the slope is steep enough the heat production rate will be larger than the heat loss rate. Therefore the temperature increases further and the stationary nonuniform state becomes unstable. Using reduced equations of motion introduced in [9], we show in the next section why this instability mechanism leads to an oscillatory instability (i.e. a Hopf bifurcation). Good candidates with a suitable resistivity function are materials which show a metal-semiconductor transition (e.g. NiS$_2$) [10]. Also some high-temperature superconductors may fall into this class [11]. In Sect. 3 these materials are modeled by a piecewise linear resistivity function. We compute stability thresholds, limit cycles, and bifurcation scenarios for this model. In Sect. 4 we show that Neumann boundary conditions (i.e. $\partial T/\partial n = 0$) may induce an oscillatory instability in a small ballast resistor (i.e. the length of the resistor is comparable to the thermal length) even though the resistivity increases monotonically with the temperature. We discuss this effect for a model where the resistivity is a step function.

2. Reduced equations of motion and the bulk instability

In this section we use two reduced equations of motion (called reduction 1 and reduction 2) introduced in [9]. In reduction 1, the state of the system is described by the temperatures of the hot and the cold domains ($T_h$ and $T_c$, resp.) and the length of the hot domain ($qL$):

$$\frac{d}{dt}T_h = -a(T_h) + j^2 \rho(T_h)$$

(2.1)

$$\frac{d}{dt}T_c = -a(T_c) + j^2 \rho(T_c)$$

(2.2)

$$\frac{d}{dt}q = \frac{1}{L} v_{hc}(j),$$

(2.3)

with

$$j(q) = \frac{E}{(1-q)\rho(T_c^0(j(q))) + q\rho(T_h^0(j(q)))},$$

(2.4)

where $d_T = \frac{d}{dT}$. Second, the width of the kinks should be small compared to the length of the domains.

The time scale of the $q$-dynamics is proportional to $L$ whereas the time scales of the $T_h, T_c$-dynamics are independent of $L$. Thus, for $L \to \infty$ (but finite $E = U/L$) $T_h$ and $T_c$ may be adiabatically eliminated leading to the following equation of motion (reduction 2):

$$\frac{dq}{dt} = \frac{1}{L} v_{hc}(j),$$

(2.6)

with

$$j(q) = \frac{E}{(1-q)\rho(T_c^0(j(q))) + q\rho(T_h^0(j(q)))},$$

(2.7)

where $T_h^0(j)$ and $T_c^0(j)$ are the equilibrium values defined by

$$a(T_h^0) = j^2 \rho(T_h^0).$$

(2.8)

In order to justify reduction 2, the states of the cold and the hot domains which are coupled through the current must be stable for fixed values of $q$ (nonlocal stability):

$$C_u = \lambda_{h,c}^0 + \lambda_{h,c}^0 - \frac{j^2}{E} [(1-q)\rho_h d_T \rho_c + q \rho_h d_T \rho_h] < 0,$$

(2.9)

$$C_{det} = \lambda_{h,c}^0 \lambda_{h,c}^0 - \frac{j^2}{E} [(1-q)\rho_h d_T \rho_c \lambda_{h,c}^0 + q \rho_h d_T \rho_h \lambda_{h,c}^0] > 0,$$

(2.10)

where

$$\rho_{h,c} = \rho(T_h^0, T_c^0), d_T \rho_{h,c} = d_T \rho(T_h^0, T_c^0).$$

(2.11)

It should be noted that the condition of local stability (i.e. $\lambda_{h,c}^0 < 0$) is always fulfilled because the stationary nonuniform states of the current-controlled ballast resistor may be interpreted as heteroclinic orbits of an undamped particle (where $x$ is the time variable and $T$ is the particle position) in the potential $\int T[a(T') + j^2 \rho(T')]dT'$ having maxima (i.e. $\lambda_{h,c}^0 < 0$) at $T_h^0$ and $T_c^0$.

Before investigating the instability, we want to understand why for metals and ordinary (i.e. not high-temperature) superconductors the stationary nonuniform state of the voltage-controlled ballast resistor is stable. All these materials have $d_T \rho \geq 0$ which implies nonlocal stability, and reduction 2 is applicable. Furthermore the stationary state of reduction 2 is stable, because $v_{hc}(j)$ is an increasing function and $d_T/dq < 0$.

The violation of condition (2.9) corresponds to a Hopf bifurcation of the system (2.1), (2.2), and (2.4) with $q \in (0, 1)$ as a control parameter. It should be noted that such an instability (i.e. $C_u = 0$ but $C_{det} > 0$) is possible only if $d_T \rho_h > 0$ and $d_T \rho_c < 0$ or vice versa. This will be