Multiple Wiener-Itô integral expansions for level-crossing-count functionals

Eric Slud*
Mathematics Department, University of Maryland, College Park, MD 20742, USA

Received June 30, 1989, in revised form May 18, 1990

Summary. This paper applies the stochastic calculus of multiple Wiener-Itô integral expansions to express the number of crossings of the mean level by a stationary (discrete- or continuous-time) Gaussian process within a fixed time interval $[0, T]$. The resulting expansions involve a class of hypergeometric functions, for which recursion and differential relations and some asymptotic properties are derived. The representation obtained for level-crossing counts is applied to prove a central limit theorem of Cuzick (1976) for level crossings in continuous time, using a general central limit theorem of Chambers and Slud (1989a) for processes expressed via multiple Wiener-Itô integral expansions in terms of a stationary Gaussian process. Analogous results are given also for discrete-time processes. This approach proves that the limiting variance is strictly positive, without additional assumptions needed by Cuzick.

1. Introduction

There is now a very well-developed stochastic calculus for smooth functionals of stochastic integrals with respect to Wiener process (Kallianpur 1980), which has been applied extensively to problems on diffusions and counting processes. This calculus could also be applied to the study of nonlinear functionals of stationary Gaussian processes, as has been remarked by Kallianpur (1980, Chap. 6), by using the representation of such functionals as multiple Wiener-Itô integral expansions. The paper of Chambers and Slud (1989b) is one effort in this direction. Indeed, since the celebrated Diagram Theorem of Dobrushin and Major (1979) can be viewed as a representation theorem for polynomials of multiple Wiener-Itô integrals, there is every hope that some nonsmooth nonlinear functionals of stationary Gaussian processes could be represented explicitly as multiple Wiener-Itô integral expansions. The nonsmooth functionals studied in this paper are the counts of level-crossings.

* Research supported by Office of Naval Research contracts N00014-86-K-0007 and N00014-89-J-1051
There are two reasons for interest in explicitly defined multiple Wiener-Itô integral expansions. First, it has been known at least since the work of Versik (1962) (see also Chap. 13 of Sinai 1977, for clarification) that spectral ergodic-theoretic properties such as weak-mixing and mixing for a functional ("factor" in the language of ergodic theory) of a stationary Gaussian process, can be expressed in terms of the absolute-continuity equivalence class of the underlying spectral measure together with the multiple Wiener-Itô integrands. A second way to exploit Wiener-Itô integral expansions has been developed by Taqqu (1975), Dobrushin and Major (1979), Maruyama (1976) and Chambers and Slud (1989a, b) among others. These authors prove general (functional) central and noncentral limit theorems for such expansions. In the present paper, central limit theorems of Chambers and Slud (1989a) will be applied to the level-crossings counts.

Our general references for multiple Wiener-Itô integrals are the monograph of Major (1981) and Chapter 6 of Kallianpur (1980). The relevant results from the general theory are summarized also in each of the papers of Chambers and Slud (1989a, b).

In this paper, \( X_t \) is a mean-0 and variance-1 stationary Gaussian process, either in discrete or continuous time as specified. Its correlation function will be denoted by \( r(t) \), and its spectral measure (assumed nonatomic) by \( \sigma \) (either on \( [\pi, \pi] \) or on \( \mathbb{R} \)). On the spaces \( L^2(\mathbb{R}^k, \sigma^k, \text{sym}) \) of complex square-integrable functions \( f \) which are symmetric in their \( k \) real arguments \( (x_1, \ldots, x_k) \) and which satisfy \( f(-x) = \overline{f(x)} \), the multiple Wiener-Itô integral operators are denoted \( I_k : L^2(\mathbb{R}^k, \sigma^k, \text{sym}) \to \mathbb{R} \), and the constant function with value 1 in the domain of \( I_k \) is denoted \( 1_k \).

2. Representation results and limit theorems

The starting point is to recognize that the indicator \( I_{[X_t, X_{t+1} < 0]} \) that \( X_t \) and \( X_{t+1} \) are of different signs, is the sum of products \( I_{[X_t < 0]} \cdot I_{[X_{t+1} > 0]} \) and \( I_{[X_t > 0]} \cdot I_{[X_{t+1} < 0]} \) of functionals which depend only on single coordinates of the underlying process \( X_t \). After expressing the functional \( I_{[X_0 > 0]} \) as a multiple Wiener-Itô integral expansion and expressing products of expansions through the Diagram Theorem, we obtain the first Proposition and Corollary.

**Proposition 1.** Define \( \rho = r(1) \), \( S_{0,0}^L = S_{1,0}^L = 1 \) for \( L \geq 1 \), and for \( j \geq 1 \)

\[
S_{L}^j = S_{L}^j(\lambda_1, \lambda_2, \ldots, \lambda_L) = \sum_{1 \leq n_1 < \ldots < n_L \leq L} e^{i(\lambda_1 + \cdots + \lambda_L)}
\]

Then

\[
I_{[X_t, X_{t+1} < 0]} = \frac{1}{2} - \frac{1}{\pi} \sum_{m=0}^{\infty} (-1)^m I_{2m} \left( \sum_{j=0}^{2m} S_{2m}^j C_{m,j}(\rho) \right),
\]

where

\[
C_{m,j}(\rho) = \sum_{\alpha \geq 0 : \alpha + j \text{ odd}} (-1)^{\alpha-1} \rho^\alpha \frac{(j + \alpha - 1)! (2m - j + \alpha - 1)!}{\alpha! \left( \frac{j + \alpha - 1}{2} \right)! \left( \frac{2m - j + \alpha - 1}{2} \right)!} 2^{-\alpha - m + 1}
\]