A SMOOTH PSEUDOCONVEX DOMAIN WITHOUT PSEUDOCONVEX EXHAUSTION

Klas Diederich and John Erik Fornaess

A pseudoconvex domain with real-analytic smooth boundary on a complex manifold is constructed which cannot be exhausted by pseudoconvex domains.

Any pseudoconvex domain \( \Omega \) on a Stein manifold \( M \) can be exhausted by pseudoconvex domains since it is itself Stein. On the other hand, it is known from examples of H. Grauert, that not any pseudoconvex domain \( \Omega \subset M \) in an arbitrary complex manifold \( M \) is holomorphically convex, even if one assumes that \( \Omega \) is real-analytic and smooth. Therefore, H. Grauert asked the question whether, nevertheless, any such domain can be exhausted by pseudoconvex domains.

In this article we want to show

**Theorem.** There exists a compact complex manifold \( M \) and a pseudoconvex domain \( \Omega \subset M \) with smooth real-analytic boundary, such that \( \Omega \) cannot be exhausted by pseudoconvex domains.

**Remark.** Notice that no assumption on the regularity of the boundaries is made for the exhausting domains.

**Construction of the Example.** Let \( H \) be the Hopf surface which is the quotient of \( \mathbb{C}^2 \setminus \{0\} \) modulo the equivalence relation generated by

\[
z \sim 2z \quad \text{on} \quad \mathbb{C}^2 \setminus \{0\}.
\]
At first, we want to define a locally trivial holomorphic $\mathbb{P}^1$-bundle $M$ over $H$, which restricts to a unit disc bundle over $H$. This restriction will then be the domain $\Omega$.

We define

$$A_1 := \{ z \in \mathbb{C}^2 : 1/2 \leq |z| < 10/12 \} \cup \{ 11/12 < |z| < 1 \}$$

$$A_2 := \{ z \in \mathbb{C}^2 : 1/2 \leq |z| < 7/12 \} \cup \{ 8/12 < |z| < 1 \}$$

and $U_1 := \pi(A_1)$, $U_2 := \pi(A_2)$, where $\pi : \mathbb{C}^2 \setminus \{0\} \rightarrow H$ is the natural projection. Then $U_1$, $U_2$ are connected, open subsets of $H$ and $U_1 \cup U_2 = H$. The intersection $U_1 \cap U_2$ consists of two connected components $V_1$, $V_2$, where $V_1$ is represented by

$$\{ z \in \mathbb{C}^2 : 8/12 < |z| < 10/12 \} \quad \text{and} \quad V_2 \text{ is represented by} \quad \{ 1/2 \leq |z| < 7/12 \} \cup \{ 11/12 < |z| < 1 \}.$$ 

Furthermore, we put for some fixed $r$, $0 < r < 1$ (close to 1),

$$T(z) := \frac{z - r}{1 - rz}.$$ 

Notice that $T$ is an automorphism of the unit disc $\Delta$ and that $T$ has two fixed points on $\partial \Delta$, namely,

$$T(1) = 1, \quad T(-1) = -1.$$ 

In order to define the bundle $M$, we denote

$$A_1 \times \mathbb{P}^1 := \{(p_1, q_1)\} \quad \text{and} \quad A_2 \times \mathbb{P}^1 := \{(p_2, q_2)\}.$$

The transition functions are given by

$$(p_1, q_1) \sim (p_2, q_2) \Rightarrow p_1 = p_2, \quad q_1 = q_2 \quad \text{for} \quad p_1 \in V_1 \quad (3)$$

$$(p_1, q_1) \sim (p_2, q_2) \Rightarrow p_1 = p_2, \quad q_1 = T(q_2) \quad \text{for} \quad p_1 \in V_2.$$ 

This defines $M \rightarrow H$ as a locally trivial holomorphic $\mathbb{P}^1$-bundle and since $T$ is an automorphism of $\Delta$, the restriction of (3) to $q_1$, $q_2 \in \Delta$ gives a disc bundle $\Sigma \subset M$, which, obviously, is a pseudoconvex domain with smooth real-analytic boundary in $M$. 