Solutions of Eliashberg equations for an electron-phonon coupling with a cutoff

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Abstract. In this paper we discuss the Eliashberg equations for the case of an electron-phonon coupling with an energy cutoff. This cutoff is imposed either for the energy difference by means of a strip function, or for both energies, with a Cooper-like expression. The strip function cutoff requires explicit calculation of not only the frequency renormalization function Z but also the energy renormalization X. The physical origin of such cutoffs might lie in the very strong electron-electron interaction which seems typical for high Tc superconductivity. If such cutoffs are admitted, the hypothesis that Tc is caused at least in part by a strong electron-phonon interaction can be reconsidered. We find that the combination of strong coupling and low-energy cutoff could produce high Tc with only small isotope effect and with little damping or pulling of the phonon modes. Correlation with other physical properties, such as specific heat, is reexamined in view to estimate the coupling constant λ. Some objections to the model using strong electron phonon interaction are removed and better agreement with observed properties is obtained.

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1 Introduction

The role of a phonon-mediated electron-electron interaction in high-temperature superconductivity is at present rather controversial. There exist rather strong arguments that seem to indicate that this interaction which is responsible for superconductivity in normal metals, is not responsible for superconductivity in the cuprates, and other “exotic” superconductors, such as organic superconductors, fullerenes, etc. The main arguments are: (a) The phonon-mediated mechanism was originally proposed because of the isotope effect $T_c \propto M^{-\alpha}$, $\alpha = 1/2$. In the cuprates, there is apparently only a very weak isotope effect [1]. (b) The Coulomb interaction suppresses “conventional” superconductivity drastically. In the cuprates, the Coulomb interaction is very strong; values of $U/(4t) \approx 3$ are frequently cited. It is difficult to obtain a $T_c$ of 120-150 K for “reasonable” values of the electron-phonon coupling constant λ with such a strong Coulomb interaction. (c) For the large values of λ required to obtain a high $T_c$, the phonon lifetime (due to absorption by conduction electrons) should be very short, and the frequency should be pulled considerably. Experimentally, such effects are weak in the cuprates [2].

The strong Coulomb interaction also causes the quasiparticle lifetime $\tau_{ee}$ to be very short. According to Landau Fermi liquid theory, $1/\tau_{ee}$ is proportional to $\xi(k)^2$, where $\xi(k)$ is the energy measured from the Fermi level. According to the marginal Fermi liquid model [3], $1/\tau_{ee}$ is proportional to $\xi(k)$. In any case, experiments indicate that $1/\tau_{ee}$ is comparable to $\xi(k)$ at energies of the order of the phonon frequency $\omega_0$ ($\approx 50$ meV in the cuprates).

Because of this short lifetime, we expect the polarization of the lattice by the electrons to be diminished. We consider here a model in which the electron-phonon coupling constant $g(k, k')$ possesses a cutoff, at values of $\xi(k)$, $\xi(k')$ of the order of the phonon frequency. We show that this cutoff modifies the interaction greatly. It reduces the effect of the Coulomb interaction drastically. It reduces, and may even eliminate, the isotope effect; and it may reduce phonon lifetime and pulling electron interaction.

2 Model

We solve the Eliashberg equations [4], with a phonon Green’s function corresponding to an Einstein spectrum, an electron-phonon coupling $g(k, k')$ corresponding either to a Cooper-like expression:

$$g(k, k') = \begin{cases} g_0 & \text{if } |\xi(k)|,|\xi(k')| < \xi_1 \\ 0 & \text{otherwise} \end{cases}$$

or a strip function:

$$g(k, k') = \begin{cases} g_0 & \text{if } |\xi(k) - \xi(k')| < \xi_1 \\ 0 & \text{otherwise} \end{cases}$$

Thus, the function $D(k, k', \omega - \omega')$ in the Eliashberg equations, along the imaginary $\omega$-axis, is given by:

$$D(k, k', \omega - \omega') = g(k, k')^2 \frac{2\omega_0}{(\omega - \omega')^2 + \omega_0^2}$$
We also consider a Coulomb interaction, which is \( \omega \)-independent (instantaneous), and may be either \( k \)-independent, as in Bogolyubov’s model \[5\], or a strip function in \( \xi - \xi' \). Thus,

\[
V(\xi, \xi') N(E_F) = \mu
\]  
\((2.4)\)

or

\[
V(\xi, \xi') N(E_F) = \begin{cases} 
\mu & \text{if } |\xi - \xi'| < \xi_2 \\
0 & \text{otherwise} 
\end{cases}
\]  
\((2.5)\)

We consider \( s \)-wave pairing, and an interaction independent of the angles of \( k, k' \). The Eliashberg equations \[6, 7\] are given by:

\[
\begin{align*}
(X(\xi, \omega(n)) - 1)\xi &= \pi T \sum_{n'} \int d\xi' \\
&\times \frac{N(\xi') V_{\text{tot}}(\xi, \xi', \omega(n) - \omega(n')) X(\xi', \omega(n')) \xi'}{\Omega} \\
(Z(\xi, \omega(n)) - 1)\omega(n) &= -\pi T \sum_{n'} \int d\xi' \\
&\times \frac{N(\xi') V_{\text{tot}}(\xi, \xi', \omega(n) - \omega(n')) Z(\xi', \omega(n')) \omega(n')}{\Omega} \\
\phi(\xi, \omega(n)) &= -\pi T \sum_{n'} \int d\xi' \\
&\times \frac{N(\xi') V_{\text{tot}}(\xi, \xi', \omega(n) - \omega(n')) \phi(\xi', \omega(n'))}{\Omega} \\
\Omega &= (Z(\xi', \omega(n')) \omega(n'))^2 + (X(\xi', \omega(n')) \xi')^2 + \phi(\xi', \omega(n'))^2
\end{align*}
\]  
\((2.6)\)

where: \( \omega(n) = \pi(2n + 1)T \), and \( V_{\text{tot}} = V - D \).

Normally, when the cutoffs \( \xi_1 \) and \( \xi_2 \) are large compared with \( \omega_0 \), the momentum renormalization function \( X \) is virtually independent of \( \phi \), i.e. the same at \( T = T_c \) and \( T = 0 \). Consequently, the momentum renormalization can be absorbed in \( \xi \), defining: \( \epsilon = \xi(1 + X) \) and replacing \( \xi \) by \( \epsilon \), and we are left with just two equations, one for \( Z \) and one for \( \phi \). Here, the cutoffs \( \xi_1 \) and \( \xi_2 \) are not assumed to be large. Therefore \( X \) turns out to depend on \( \phi \), and be different at \( T = T_c \) and \( T = 0 \) \[7\]. Therefore it cannot be simply absorbed in the single-particle energy, and we must deal with the three coupled equations for \( X, Z \) and \( \phi \). We deal mainly with the \( T = 0 \) equations, but also solve the temperature dependence in a few cases. The integration over \( \xi' \) was carried out over an interval

\[-20\omega_0 \leq \xi' \leq 20\omega_0\]

with a step size of \( d\xi' = 0.4\omega_0 \) (in a few cases, \( 0.2\omega_0 \)). The sum over the Matsubara frequencies was carried out in most cases over 12 frequencies. At low temperatures, a larger number was necessary to attain convergence. The \( T = 0 \) results were obtained by extrapolating the finite (but low) \( T \) results to \( T = 0 \).

3 Results

3.1 No Coulomb interaction

In Fig. 1 (full line), we plot the gap function \( \Delta = \phi/Z \) at \( \xi = 0, n = 0 \) at \( T = 0 \) as function of the cutoff \( \xi_1 \), for both the Cooper-like \( g(k, k') \), and the strip form, for \( \lambda = g_0^2 N(E_F) = 2 \). We see that \( \Delta \) is rather large even for a cutoff \( \xi_1 = \omega_0 \). When \( \xi_1 \) is considerably smaller than \( \omega_0 \), \( \Delta \) falls linearly with \( \xi_1 \), as expected.

3.2 Constant Coulomb interaction

In Fig. 1 (broken line), we plot \( \Delta \) as function of the cutoff \( \xi_1 \), for a \( k \)-independent Coulomb repulsion with \( \mu = 1 \) (\( \lambda = 2 \) as before). We see that as the cutoff \( \xi_1 \) decreases, \( \Delta \) actually increases, and by a large factor. For a “normal” electron-phonon coupling (i.e. without cutoff), the Coulomb interaction reduces \( \Delta \) by about a factor of 3. For a cutoff \( \xi_1 = \omega_0 \), there is almost no reduction (\( \approx 20\% \) for the Cooper-like interaction, \( \approx 10\% \) for the strip interaction). Thus, the cutoff diminishes the effect of the Coulomb interaction very strongly.

In Fig. 2 we plot the functions \( \phi, Z, X \), and the excitation energy \( E = \sqrt{\phi^2 + (X\xi)^2 / \Omega} \) as function of \( \xi \) for the Cooper-like and strip interactions, for \( \lambda = 5, \mu = 1 \). For a larger value of \( \mu \), a solution that is antisymmetric in \( \xi \) (Fig. 2 (broken line)) becomes more stable (it is still symmetric in \( \omega \)). In Fig. 3 we also show a 3D plot of \( \phi(\xi, i\omega) \), as function of \( \xi \) and \( i\omega \), for both the symmetric (a) and antisymmetric (b) solutions.

The temperature-dependence of the gap functions \( \Delta_{\text{max}} \) is shown in Fig. 4, for several values of the parameters. The temperature-dependence is approximately BCS-like. The ratio \( 2\Delta_{\text{max}}(T = 0)/T_c \) has values characteristic of strong coupling (\( 6.5 \pm 0.3 \)).