High-temperature series expansion for the relaxation times of the two dimensional Ising model

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Abstract. We derive the high temperature series expansions for the two relaxation times of the single spin-flip kinetic Ising model on the square lattice. The series for the linear relaxation time $\tau_l$ is obtained with 20 non-trivial terms, and the analysis yields $2.183 \pm 0.005$ as the value of the critical exponent $\nu_l$, which is equal to the dynamical critical exponent $\nu$ in the two-dimensional case. For the non-linear relaxation time we obtain 15 non-trivial terms, and the analysis leads to the result $\nu_{nl} = 2.08 \pm 0.07$. The scaling relation $\nu_l - \nu_{nl} = \beta$ (\(\beta\) being the exponent of the order parameter) seems to be fulfilled, though the error bars of $\nu_{nl}$ are still quite substantial. In addition, we obtain the series expansion of the linear relaxation time on the honeycomb lattice with 22 non-trivial terms. The result for the critical exponent is close to the value obtained on the square lattice, which is expected from universality.

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1. Introduction

In the last several years there has been an increasing interest in the determination of the dynamical critical exponent $\nu$ of the Ising model, which describes the effect of critical slowing down while approaching the critical point $T_c$. Although the Ising model is probably the best studied model in statistical physics and critical phenomena – it can be solved exactly in two dimensions [1] – there is a variety of results and conjectures about the dynamical critical exponent $\nu$. The only consensus seems to be, that the relaxation time of the model diverges like $|T - T_c|^\nu$, or expressed in terms of the correlation length like $\xi^\nu$ [2].

As the Ising model is an intrinsically static model, one has the freedom to introduce dynamics, as long as some physical constraints are fulfilled, e.g. the right description of equilibrium phenomena. From that point of view it is obvious, that different dynamical schemes have different dynamical exponents $\nu$, though they lead to the same equilibrium and they describe the same statistical properties of the model. This has been used recently in the development of new Monte Carlo techniques – the Monte Carlo method is probing the phase space by “randomly” chosen states, which can be also interpreted dynamically – that in certain cases do not suffer from critical slowing down, which means that the value of $\nu$ is quite small or even zero [3, 4]. These new techniques are non-local, in the sense that they allow to go from one state to another by changing more than one local state (spin flip), while traditional Monte Carlo techniques use local updating schemes (e.g. single spin-flip dynamics). In this paper we discuss the kinetic Ising model with single spin flip dynamics, as introduced by Glauber [5], which is in the dynamical universality class of the local update algorithms.

The values for the corresponding dynamical critical exponent $\nu$ in two dimensions in the literature vary from $\nu = 1.82$ [6] to $\nu = 2.34$ [7] (or even higher or lower values), obtained with a variety of methods ranging from Renormalization Group techniques, High Temperature Series Expansions to Monte Carlo methods and experiments. For recent compilations of the results we refer to [8] and [9]. There has also been a recent conjecture by Alexandrowicz [10], that the dynamical critical exponent can be expressed in terms of static exponents and a new “geometrical exponent”, which is suggested to be of static origin. It is a rather unsatisfactory situation that the static exponents of the Ising model are known exactly [11] (in two dimensions), or with very high accuracy [12] (in three dimensions), but there is no consensus about the value of the dynamical critical exponent yet. In one and in four dimensions the exponent is known to be exactly 2, and there is also a conjecture by Domany [13], that $\nu = 2$ also in two dimensions, with possibly logarithmic corrections.
In this work we use a high temperature series expansion method to determine $z$ (parts of the results have been published elsewhere [16]). The method was developed by Suzuki and Yahata [14, 15] more than twenty years ago, and applied to the same problem as presented in this work, though on the square lattice only. It has been also applied to study the dynamics of spin glasses [17, 18], and to calculate the auto-correlation time [7, 19]. In Sect. 2 we present a short introduction to the kinetic Ising model, and how one can calculate time dependent expectation values. Section 3 deals with the method we use for the high temperature expansion of the relaxation times. We present also an interpretation of the series expansion in terms of graphs. In Sect. 4 we give a short overview on the computational techniques we have used, as well as some of the rules to reduce the combinatorial explosion of the problem. The analysis of the series obtained is presented in Sect. 5, where we also describe the used analysis methods, as necessary. Finally in Sect. 6 we summarize and discuss our results.

2. The model

In this section we summarize some aspects of the kinetic Ising model. For a more detailed description we refer to the [2, 5, 15, 20, 22, 25]. The static Ising model with an external magnetic field $H$ is described by the Hamiltonian

$$
\mathcal{H} = -J \sum_{\langle i,j \rangle} s_i s_j - \mu H \sum_i s_i, \quad J > 0, s_i = \pm 1.
$$

(2.1)

The dynamics is introduced by coupling the system to a heat bath. This leads to stochastic spin-flips $s_i \rightarrow -s_i$ with transition rates $W_i(s_i)$, that are chosen such as to fulfill the condition of “detailed balance”, e.g. [5]

$$
W_i(s_i) = \frac{1}{2} [1 - s_i \tanh (\beta E_i)].
$$

(2.2)

$E_i = J \sum_{k} s_k + \mu H$ is the local field at site $i$ (the sum is running over all nearest neighbours of spin $s_i$). It should be noted, however, that there are other possibilities of choosing the transition rates [5, 20].

The time development of the system can be described by a master equation. Using the formal solution of this equation, one can calculate time dependent expectation values [22], e.g. $A(t) = \langle e^{-\lambda t} A \rangle$. $\langle \ldots \rangle$ denotes the equilibrium expectation value, and

$$
\lambda(t) = \sum_i W_i(s_i)(1 - P_i)
$$

(2.3)

is the Liouville operator ($P_i$ is the spin flip operator of spin $s_i$).

The linear relaxation time $\tau_1$ can be derived by expanding the dynamic susceptibility [14, 15]

$$
\chi(\omega) = \frac{1}{N} \sum_{i,j} \left< \frac{\mathcal{L}}{\mathcal{L} + i\omega} s_i s_j \right>
$$

(2.4)

for low frequencies $\omega$ (a factor $\beta$ is absorbed in the definition of $\chi$ as usual): $\chi(\omega) = \chi(0) [1 - i\omega \tau_1 + \mathcal{O}(\omega^2)]$. $\chi(0)$ is the static susceptibility, and

$$
\tau_1 = \chi(0)^{-1} \sum_{i,j} \left< s_i \mathcal{L}^{-1} s_j \right>
$$

(2.5)

is the linear relaxation time. For convenience we define:

$$
\tau_1 := \chi(0) \tau_1 = \frac{1}{N} \sum_{i,j} \left< s_i \mathcal{L}^{-1} s_j \right>.
$$

(2.6)

The non-linear relaxation time is derived by a similar calculation from the time dependent magnetisation [22, 23], in the case of a non-zero equilibrium magnetisation:

$$
\tau_2 = \frac{1}{N} \sum_i \left< \mathcal{L}^{-1} s_i \right>,
$$

(2.7)

$\langle \ldots \rangle_a$ denotes the equilibrium expectation value with $\langle M \rangle_a \neq 0$. For the sake of simplicity this is chosen to be the fully magnetised state with all spins up.

At the critical temperature $T_c$ the relaxation times diverge as $\tau_c = |T - T_c|^{\beta}$. $T_c$ is known exactly for the two-dimensional Ising model [1], and the relation $\lambda_2 = \lambda_1 - \beta$ between the exponents is known from scaling theory (for a detailed description see [23] and [24]). $\beta$ is the static exponent of the magnetisation, which is also known exactly ($\beta = 0.125$). It should be noted that the value of $\beta$ is small compared to the values we expect for the two other exponents.

3. The method

The calculation of high-temperature series of the relaxation times can be divided into two parts: First the application of $\mathcal{L}^{-1}$ on a spin $s_i$, which leads to multi-spin-clusters, and – as a second step – the computation of the equilibrium expectation values of multi-spin-correlations, resulting of step one. We follow the work of Suzuki and Yahata [14, 15] in the derivation of the operators.

3.1. First step

Before we derive the expansion of $\mathcal{L}^{-1}$ we have to take a look at the operator $\mathcal{L}$ first. The temperature dependence of $\mathcal{L}$ is located in the transition probability, which in the zero field case is

$$
w_k(s_k) = W_k(s_k)_{|\mu=0} = \frac{1}{2} \left[ 1 - s_k \tanh \left( \frac{K \sum_i (k) s_i}{2} \right) \right],
$$

(3.1)

where $K = \beta J$ The following identity is valid for three and four nearest neighbours, that is for the honeycomb and the square lattice, respectively:

$$
tanh \left( \frac{K \sum_i (k) s_i}{2} \right) = c_1 \sum_i (k) s_i + c_3 \sum_{i,m,n} (k) s_i s_m s_n,
$$

(3.2)

where $\sum_{i,m,n}$ denotes a sum over all possible triples of nearest neighbours. Eq. (3.2) can be obtained in different ways, e.g. by using it as an ansatz, and evaluating it for all