Exact Solution of a Langevin Equation with Nonlinear Periodic Noise

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An exact stochastic average of a Langevin equation with a multiplicative nonlinear periodic noise is performed. The noise is described by an arbitrary periodic function of the diffusion Wiener-Lévy stochastic process. The solution of this stochastic equation is given by periodic solutions of the Hill equation.

I. Introduction

The dynamics of many realistic systems can be described by the following rate equation:

\[ \frac{d\rho}{dt} = R \cdot \rho \]  

(1.1)

where \( \rho \) is the dynamical variable of the system and \( R \) is the appropriate rate. When the system is an open system which is subject to an external noise \( x(t) \), the rate \( R \) may become a function of the noise itself. In a recent paper [1] an exact solution for the stochastic average of \( \rho \) has been obtained for the case when \( R \) depends on a linear and quadratic Gaussian noise modeled by the Ornstein-Uhlenbeck stochastic process. Many problems in Quantum Optics with an external driving laser light lead to such rate equations. The source of noise is mainly due to the partial coherence of the laser. Gaussian amplitude fluctuations in one-photon effects lead to a rate equation with a quadratic noise [2].

But in many realistic applications the fluctuating phase of the field becomes important too and in such the rate \( R \) depends on the fluctuating phase and is a periodic function in the phase. In laser physics it is well known that phase fluctuations are very well described by the Wiener-Lévy stochastic process [3, 4].

These physical applications motivate the investigation of the stochastic equation (Langevin Eq. (1.1)) with the rate \( R \) being an arbitrary periodic function of the Wiener-Lévy stochastic process \( x(t) \). In this paper we show that an exact equation for the stochastic expectation value of \( \rho \) can be obtained. The dynamics of \( \rho \) for an arbitrary function \( R \) is given by periodic solutions of the Hill equation. This Hill equation plays the role of a Fokker-Planck equation for the stochastic equation (1.1). In Sect. II we derive this Fokker-Planck type equation and next discuss briefly some of its properties. In Sect. III we present a closed form solution for the stochastic expectation value of \( \rho \) in the case when the rate \( R \) is described by a finite number of harmonics. Following the method of vector-tridiagonal recurrence relation studied recently [5] we write the closed-form solution in terms of a continued fraction. This form of the solution is particularly well suited for numerical investigations.

II. Exact Stochastic Average of \( \rho \)

In order to investigate the dynamical evolution of the stochastic expectation value of \( \rho \) we introduce the following generating function:

\[ g_n(t) = \exp \left( i n \frac{2\pi}{T} x(t) \right) \rho(t) \quad \text{for} \quad n = 0, \pm 1, \pm 2, \ldots \]  

(2.1)

From the Langevin equation (1.1) we derive the following equation for \( g_n \):

\[ \frac{\partial}{\partial t} g_n(t) = i n \frac{2\pi}{T} x \cdot g_n(t) + \sum_{k=-\infty}^{\infty} b_k g_{n+k} \]  

(2.2)

where the coefficients \( b_k \) are given by a Fourier expansion of the function \( R \).
\[ R = \sum_{k=\infty}^{k=\infty} b_k e^{i k \frac{2 \pi}{T} \xi}. \]  (2.3a)

The deterministic value of the rate i.e. without noise is given by

\[ R_d = \sum_{k=\infty}^{k=\infty} b_k \]  (2.3b)

and depends crucially on the form of the \( b_k \) coefficients. We shall assume that the deterministic rate \( R_d \) exists i.e. the series (2.3b) is convergent. The recurrence relation (2.2) with a stochastic multiplicative noise \( \xi \) can be written in the following infinite dimensional matrix form:

\[ \frac{\partial \mathbf{g}}{\partial t} = (M_0 + i \mathbf{M}) \mathbf{g} \]  (2.4)

where the form of the infinite matrices \( M_0 \) and \( M \) and of the vector \( \mathbf{g} \) is clear from (2.2). It is well known that the derivative of the Wiener-Lévy stochastic process entering (2.4) is a Gaussian white noise with mean value equal to zero and covariance:

\[ \langle \dot{\mathbf{x}}(t) \dot{\mathbf{x}}(s) \rangle = 2 \Gamma \delta(t-s) \]  (2.5)

where \( \Gamma \) is the diffusion constant of the Wiener-Lévy process.

The operator stochastic equation (2.4) with a multiplicative stochastic white noise can be averaged exactly if the initial condition is independent on the stochastic properties of the random variable \( \dot{x} \). In our case \( g_0(0) \) depends on \( x(0) \). Nevertheless it has been shown [4] that due to the properties of the Wiener-Lévy stochastic process this initial phase dependence is irrelevant for the form of the exact stochastic average of (2.4) with a Gaussian noise \( \dot{x} \) given by (2.5). As a result we obtain:

\[ \frac{\partial \mathbf{g}}{\partial t} = (M_0 - \Gamma \mathbf{M}^2) \mathbf{g} \]  (2.6)

which is equivalent to the following recurrence relation

\[ \frac{\partial}{\partial t} \langle g_n \rangle = - \Gamma n^2 \left( \frac{2 \pi}{T} \right)^2 \langle g_n \rangle + \sum_{k=\infty}^{k=\infty} b_k \langle g_{n+k} \rangle \]  (2.7)

with the initial condition \( \langle g_n(0) \rangle = \rho(0) \) for \( n = 0, \pm 1, \pm 2, \ldots \). This infinite recurrence relation can be transformed into a differential equation with the help of the following function:

\[ G(t, \phi) = \sum_{n=\infty}^{n=\infty} \langle g_n \rangle e^{i n \frac{2 \pi}{T} \phi} \text{ where } \phi \in \mathbb{R}. \]  (2.8)

We obtain from (2.7) the following differential equation for \( G(t, \phi) \):

\[ \frac{\partial}{\partial t} G(t, \phi) = \left( \Gamma \frac{\partial^2}{\partial \phi^2} + R(\phi) \right) G(t, \phi) \]  (2.9)

with the initial condition

\[ G(0, \phi) = \sum_{n=\infty}^{n=\infty} g_0(0) e^{i n \frac{2 \pi}{T} \phi} = 2 \pi \rho(0) \sum_{n=\infty}^{n=\infty} \delta \left( \frac{2 \pi \phi}{T} + 2 \pi n \right) \]  (2.10)

and periodic boundary condition

\[ G(t, \phi + T) = G(t, \phi). \]  (2.11)

Equation (2.9) plays the role of a Fokker-Planck equation for the dynamical variable \( \rho \) with the difference that the expectation value of \( \rho \) is given by the following Fourier-form formula

\[ \langle \rho(t) \rangle = \frac{1}{T} \frac{d \phi}{0} G(t, \phi). \]  (2.12)

It is clear from the structure of (1.1) and (2.9) that all the moments of \( \rho \) can be calculated in the same way just performing the following transformation \( \langle \rho \rangle \rightarrow \langle \rho^\prime \rangle \) and \( R \rightarrow \Gamma, R \) in all our equations of this section.

The function \( G(t, \phi) \) plays the role of a Green’s function for the equation (2.9) and accordingly we can write it in the following form:

\[ G(t, \phi; 0, 0) = \sum \psi_j(\phi) e^{\lambda_j t} \psi_j^*(0) \]  (2.13)

where \( \psi_j(\phi) \) and \( \lambda_j \) are eigenfunctions and eigenvalues of the following equation:

\[ \left( \Gamma \frac{\partial^2}{\partial \phi^2} + R(\phi) - \lambda_j \right) \psi_j(\phi) = 0 \]  (2.14)

with periodic boundary condition. If the eigenvalue problem (2.14) is not self-adjoint we have to be careful and it is necessary than to distinguish the left and the right eigenvalues. We recognize in the equation (2.14) the Hill equation and \( \lambda_j \) are proper roots of the Hill’s determinant [8].

The stochastic expectation value of \( \rho(t) \) can be written according to (2.12) in the following form

\[ \langle \rho(t) \rangle = \sum \psi_j^*(0) \psi_j(0) e^{\lambda_j t}. \]  (2.15)

The solution of the dynamical problem for the stochastic expectation value of \( \rho(t) \) is given by eigenstates and eigenvalues of periodic solutions of the