Products of Random Variables Depending on a Random Walk

By

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Dedicated to Prof. Dr. L. Schmetterer on his 60th Birthday

Abstract. Let $X = (X_n)_{n \geq 0}$ denote an irreducible random walk ("ergodic" in the sense of [7]) on a compact metrizable abelian group $G$. In this paper we characterize completely the limit distributions of the products $Y_n = X_0 \cdots X_n$. In particular we find necessary and sufficient conditions for $X$ and/or $G$ to imply that the products are asymptotically equidistributed in the mean, i.e.

$$\lim_{n \to \infty} \left( \frac{1}{n} \sum_{k=1}^{n} (Y_k \in A \mid Y_0 = y) \right) = m_G(A)$$

holds for all open, $m_G$-regular subsets $A$ of $G$ ($m_G$: normalized Haar measure). For example if $G$ is monothetic and connected or if $X$ is asymptotically equidistributed (not merely in the mean) then the products are asymptotically equidistributed in the mean.

Introduction

Let $G$ denote a compact abelian metrizable group and let $\mu$ be a Radon probability measure on $G$ the support supp($\mu$) of which generates $G$. Let $X = (X_n)_{n \geq 0}$ denote a random walk on $G$ with distribution $\mu$, i.e. such that $\text{Prob}(X_{n+1} \in A \mid X_n = x) = \mu(x^{-1}A)$ holds for all $x \in G$ and for all Baire subsets $A \subset G$. It is wellknown that $X$ is "asymptotically equidistributed in the mean" in the following sense: if $m_G$ denotes the normalized Haar measure on $G$, and if $A$ is an open subset satisfying $m_G(A) = m_G(\bar{A})$ ($\bar{A}$: closure of $A$) then

$$\lim_{n \to \infty} \left( \frac{1}{n} \sum_{k=1}^{n} \text{Prob}(X_k \in A \mid X_0 = x) \right) = m_G(A).$$

Moreover there are well-known conditions for $\mu$ to imply the stronger limit theorem

$$\lim_{n \to \infty} \text{Prob}(X_n \in A \mid X_0 = x) = m_G(A).$$
In this latter case we shall call \( X \) to be \textit{asymptotic equidistributed}. (For results more general than these see [2, 7].)

It is the aim of the present paper to determine the limit distributions of the products \( Y_n = X_0 X_1 \ldots X_n \) for a given random walk \( (X_n)_{n \geq 0} = X \). In particular we give necessary and sufficient conditions for \( X \) and/or \( G \) to imply that \( Y = (Y_n)_{n \geq 0} \) is asymptotically equidistributed in the mean. In addition we present a sufficient condition on \( X \) to force the products to be asymptotically equidistributed.

Like in the finite case the main tool is the study of the associated Koutsky process \( Z = (X, Y) \) on \( G \times G \). The classification of the states of this latter process follows from the more general theory developed in [12] which is highly inspired by the work of L. Schmetterer who treated the finite case (see e.g. [11]).

The limit distributions of \( Z \) (and hence of \( Y \)) are described in sect. 3. Note that the general theorem 4.8 of [12] is not applicable in the present context since in general the random walk \( X \) is not uniformly ergodic in the sense used there.

The various criteria for \( Y \) to be asymptotic equidistributed (in the mean) are presented in the fourth section (4.4 and 4.9 to 4.12).

The first two sections contain the basic facts and definitions to make the paper more readable and self-contained. In particular sect. 2 contains a summary of some important results about the process \( Z \) taken from [12].

Finally let us point out that the results are easily extendable to the case of an abelian semigroup. The corresponding theory for non-abelian groups seems to be substantially more complicated and the methods of Fourier analysis used here are not suitable in the general case.

\section{Preliminaries}

\subsection{1.1.}
Let \( K \) denote a compact metric space. We consider only temporally homogeneous Markov chains \( X = (X_n)_{n \geq 0} \) with state space \( K \), where the associated Markov kernel \( P \) is Feller (continuous in [5], p. 175), i.e. for every \( f \in \mathcal{C}(X) \) (the space of continuous complex valued functions on \( X \)) the function \( T_P f \), defined by

\[ T_P f(x) = \int f(y) P(x, dy) = E(f \circ X_n \mid X_{n-1} = x) \]

is continuous. Thus there is an obvious bijection between such