On the Radicals of Structural Matrix Rings

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Abstract. The relationship between the radical of a ring \( R \) and a structural matrix ring over \( R \) has been determined for some radicals. We continue these investigations, amongst others, determining exactly which radicals \( \gamma \) have the property \( \gamma(M(\rho, R)) = M(\rho, \gamma(R)) + M(\rho, \gamma^+(R)) \) for any structural matrix ring \( M(\rho, R) \) and finding \( \beta(M(\rho, R)) \) for any hereditary subidempotent radical \( \beta \).

Structural matrix rings as a generalization of full matrix rings and upper (or lower) triangular matrix rings (cf. VAN WYK [7], [8]), provide a useful class of examples (or counterexamples) of rings. The relationship between the radical of a ring \( R \) and the radical of a structural matrix ring over \( R \) can vary considerably, often because of the presence of a certain nilpotent ideal of the structural matrix ring. It seems that this makes it impossible to give a canonical description of the radical of the structural matrix ring in terms of the radical of the base ring as is the case for full matrix rings. The relationship between the radical of the ring and the radical of the structural matrix ring relies in no insignificant way on the type of radical under consideration (e.g. hypernilpotent or subidempotent) and, especially in the case of subidempotent radicals, on the form of the structural matrix ring. This relationship has been determined for special radicals (VAN WYK [9]), subsequently for a wider class of radicals which includes the normal radicals (SANDS [6]) and recently, for some subidempotent radicals (GROENEWALD and VAN WYK [4]). We continue these investigations here. SANDS [6] has shown that for radicals \( \gamma \) with the matrix property and for which \( R \in \gamma \) implies \( R^0 \in \gamma \), the relationship between the radical of a ring \( R \) and that of a structural matrix ring \( M(\rho, R) \) over

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$R$, is given by
\[
\gamma(M(\rho, R)) = M(\rho_s, \gamma(R)) + M(\rho_a, \gamma^+(R))
\] (terminology is explained below). We will show (Theorem 1) that Sands chose wisely: the radicals he considered are precisely the only radicals for which this relationship holds. We also give a canonical description of $\gamma(M(\rho, R))$ for any hereditary subidempotent radical $\gamma$ with the matrix property (Theorem 11).

Rings will be associative; not necessarily with identity. Radicals considered will be Kurosh-Amitsur radicals. For standard definitions and results WIEGANDT [10] can be consulted. In our description of structural matrix rings, we follow the approach of SANDS [6].

Let $I$ be a finite index set with cardinality $|I| = n$. Let $\rho$ be a transitive relation on $I$. For any ring $R$, the set $M(\rho, R)$ of all $n \times n$ matrices $A = [a_{ij}]$ over $R$ with $a_{ij} = 0$ for all $(i, j) \in I \times I$ with $(i, j) \notin \rho$, forms a subring of $M_n(R)$, the full $n \times n$ matrix ring over $R$. $M(\rho, R)$ is called the structural matrix ring over $R$ determined by $\rho$. In [7, 8, 9] VAN WYK requires the relation $\rho$ to be also reflexive, but this is not necessary to obtain a subring of $M_n(R)$. Let $\rho_s := \{(i, j) \in \rho | (j, j) \in \rho\}$ and $\rho_a := \{(i, j) \in \rho | (j, i) \notin \rho\}$. Let $J = \{j \in I | (j, j) \in \rho\}$. Then $\rho_s \subseteq J \times J$, $\rho_s$ restricted to $J$ is an equivalence relation and $\rho_a$ is a transitive relation. As an additive group, $M(\rho, R)$ is the direct sum of $M(\rho_s, R)$ and $M(\rho_a, R)$. $M(\rho, R)$ is a nilpotent ideal of $M(\rho, R)$ (with degree of nilpotency at most $n$) and $M(\rho_s, R)$ is a subring of $M(\rho, R)$. As $\rho_s$ is an equivalence relation on $J$, there are disjoint equivalence classes $J_1, J_2, \ldots, J_m$ with $\rho_s = \bigcup_{t=1}^{m} \rho_t$ where $\rho_t = J_t \times J_t$. Hence $M(\rho_s, R)$ is a direct sum of full matrix rings $M_n(R)$ where $n_t = |J_t|$. As a subring of $M(\rho, R)$, $M_n(R) = M(\rho_t, R)$ and each $A = [a_{ij}] \in M(\rho_t, R)$ has $a_{ij} = 0$ unless $(i, j) \in \rho_t$. Each $M(\rho_t, R)$ is an ideal of $M(\rho_s, R)$, but not necessarily of $M(\rho, R)$.

It will be convenient to further decompose the above decomposition of $M(\rho, R)$ as a sum of the symmetric part $M(\rho_s, R)$ and the antisymmetric part $M(\rho_a, R)$ as follows:
\[
M(\rho, R) = M(\rho_s, R) + M(\rho_a, R) = \bigoplus_{t=1}^{m} M(\rho_t, R) + M(\rho_a, R) =
\]
\[
= (K \oplus L) + M(\rho_a, R) = K \oplus (L + M(\rho_a, R))
\]