Stability Domain of Coherent Laser Waves

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The constant-amplitude solutions with wavelength $A$ of a semiclassical laser model exhibit three different instabilities in the $(\lambda, A)$-plane for pump parameters $\lambda$ above threshold.

Many spatially extended nonequilibrium systems are known where, upon changing a control parameter bifurcation to a whole family of new solutions occurs. Often many of these are linearly stable and the state of the system is hence not uniquely determined above the instability threshold [1-6]. In this note we investigate the laser-threshold from this point of view. Recently there has been a lively interest [7, 8] in the study of bifurcation problems in a laser model, based on semiclassical equations as derived for instance by Risken and Numedal [9]. In their nomenclature and normalisation these equations read in the "rotating wave approximation"

$$
P + \gamma_{\perp} P = \gamma_{\|} E \sigma, $$
$$
\dot{\sigma} + \gamma_{\|} \sigma = \gamma_{\perp} [\lambda + 1 - \frac{1}{2} \lambda (E^* P + EP^*)], $$
$$
E + c \frac{\delta E}{\delta x} + \kappa E = \kappa P, $$

(1)

where $P$ is the (normalized) polarisation, $E$ the electric field, $\sigma$ the population inversion of the active medium. The constants $\gamma_{\perp}$ and $\gamma_{\|}$ are relaxation rates for atomic polarization and population inversion respectively while $\kappa$ describes the cavity losses and $c$ is the velocity of light. The pumping parameter $\lambda$ is always positive and $\lambda = 0$ is the threshold for laser action. In (1) no spatial dependence perpendicular to the direction of light propagation occurs and a single direction of polarization is assumed.

We concentrate on the investigation of the coherent wave solutions (cw) of (1), which can be parametrized by the phase difference $\varphi$ between $E$ and $P$

$$
E = (1 - \tan^2 \varphi / \lambda)^{1/2} \exp \left[ (q_0 x - \omega_0 t) + i \varphi \right],
$$
$$
P = E e^{-i\varphi / \cos \varphi},
$$
$$
\sigma = \cos^{-2} \varphi,
$$
$$
\omega_0 = \gamma_{\perp} \tan \varphi, \quad c q_0 = (\gamma_{\perp} + \kappa) \tan \varphi, \quad \tan^2 \varphi \leq \lambda,
$$

(2)

where $\varphi$ is an arbitrary constant. These solutions bifurcate continuously out of the unstable nonlasing state as $\lambda$ increases from zero to infinity. On the bifurcation line $\lambda_b = \tan^2 \varphi$ (see Fig 1) the states have zero amplitude.

The small amplitude solutions in the neighbourhood of this bifurcation line are unstable for $\varphi + 0$. This is because for $\lambda > 0$ the nonlasing state $E = P = 0$ has exponentially growing modes and this remains true for the neighbouring small amplitude solutions for continuity reasons. On the other hand for the $\varphi = 0$-solution Risken and Numedal [9] found stability up to a critical value

$$
\lambda = \lambda_{\text{cr}}(\varphi = 0) \equiv \lambda_1 = 4 + 3 r + 2 \sqrt{4 + 6 r + 2 r^2},
$$
$$
r = \gamma_{\perp} / \gamma_{\|}. $$

(3)

Of course one expects for continuity reasons that the neighbouring states with small enough $\varphi$-values are also stable in this range of $\lambda$. It is the purpose of the present note to investigate the entire region of linear stability of these solutions in the $(\lambda, \varphi)$-plane. This question is certainly of interest when one tries to write down simpler equations than (1) which describe the behaviour of the system for $\lambda$-values near the thresholds $\lambda_1$ [7] and zero [10].
expression but since we are here interested in the stability limits we look for parameter values for which real solutions \( \omega(q) \) exist for some \( q \). In this case one can achieve a simplification by introducing the variable \( x = (\omega - c_0 q)/\kappa \) instead of \( q \) and \( \tilde{\omega} = \omega/\gamma_\perp \). In these variables equation (6) reduces to a third order polynomial in \( \tilde{\omega} \) which reads

\[
\tilde{\omega}^3 \left[ -2x + i(x^2 - 1 - \Phi) \right] \\
+ \tilde{\omega}^2 \left[ x(1 + \Phi) - x^2(2 + r) - i2x(1 + r - \Phi) \right] \\
+ i\tilde{\omega} \left[ (\lambda + 1)r x^2(1 + \Phi) + i2\lambda x(1 + \Phi) = 0 \right],
\]

where \( \Phi = \tan^2 \varphi \) and \( \lambda = \lambda \cos^2 \varphi - \sin^2 \varphi \). It is remarkable that the cavity-loss parameter \( \kappa \) does not enter this equation and hence that all marginal frequencies and critical parameters \( \lambda_\varphi(x_\varphi) \) are independent of \( \kappa \).

The first instability mechanism we want to discuss occurs in the hydrodynamic branch. The fact that the phase \( \chi \) in (2) can be chosen arbitrarily leads to a phase diffusion mode with a dispersion law

\[
\omega(q) = \nu q - iD q^2 + O(q^3).
\]

Looking for such a solution of (6) in a small-\( q \) expansion yields the diffusion constant

\[
D = \frac{\gamma_\perp e^2 \kappa}{(\gamma_\perp + \kappa)^3} \frac{1 - 2\tan^2 \varphi/\lambda}{1 + \tan^2 \varphi}.
\]

Furthermore it is of interest to see what type of instabilities limit the region of stable stationary solutions.

The stability of the solutions (2) is tested by perturbing the amplitudes and phases by small amounts \( \delta E(x, t), \delta P(x, t), \delta \sigma(x, t), \delta \chi(x, t), \delta \varphi(x, t) = \delta u(x, t) \) and then by linearizing the equations (1) in these perturbations. Since the resulting equations are uniform in space and time a plane-wave ansatz

\[
\delta u = \delta u_\varphi e^{i(q(x - \omega t))} 
\]

transforms the linear differential equations into an algebraic problem

\[
L(\omega, q) \cdot \delta u_\varphi = 0.
\]

The five dispersion laws \( \omega(q) \) are then found by solving the fifth-order polynomial in \( \omega \)

\[
\det [L(\omega, q)] = 0.
\]

A solution (2) is stable when \( \text{Im}[\omega(q)] \leq 0 \) for all \( q \)-values. The polynomial (6) is a fairly lengthy ex-

Table 1. Points of instabilities of coherent waves for \( r = 0.5 \)

<table>
<thead>
<tr>
<th>( \lambda_\varphi )</th>
<th>( \gamma_\varphi )</th>
<th>( x_\varphi )</th>
<th>( \omega_\varphi )</th>
<th>Instability</th>
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