Stability of Nonlinear Diffusion

W. Kern and B.U. Felderhof
Institut für Theoretische Physik A, RWTH Aachen, Aachen, Germany

Received August 5, 1977

The nonlinear diffusion equation in bounded geometry with time-independent boundary conditions has a uniquely determined stationary solution. We show that this solution is dynamically stable in the sense of Liapunov. Any initial distribution tends to the stationary one as time goes on. It is shown that the application of the Glansdorff-Prigogine stability criterion requires a more elaborate analysis. We develop a variational procedure which has application in a wide range of nonlinear transport problems.

1. Introduction

In many diffusion problems the diffusion coefficient depends nonlinearly on the local density of the diffusing substance [1–4]. As examples we mention fluid flow through a porous medium, heat conduction in shells and rockets, transmission line pulses, etc. For bounded geometry with time-independent boundary conditions the nonlinear diffusion equation has a uniquely determined stationary solution, as one easily shows by a transformation of variable and an application of Dirichlet's principle. The solution of the initial-value problem, with possibly time-dependent boundary conditions has been shown to be unique by Weiss [5]. In this article we show that the stationary solution for time-independent boundary conditions is asymptotically stable in the sense of Liapunov. This implies that an arbitrary initial distribution tends to the stationary one in the course of time.

We compare the analysis with an application of the Glansdorff-Prigogine criterion [6]. This criterion is based on thermodynamic considerations and states that a sufficient condition for stability of the stationary state against small perturbations is that the rate of change of the second order deviation of the entropy from its steady state value is positive. The analysis required for this criterion is more elaborate and less rigorous. A modified version of this criterion involving a suitable weighting function in the second order entropy leads to a proof of local stability.

In Section 2 we formulate the basic equations. In Section 3 we prove the dynamical stability of the stationary state, making use of a Liapunov function and the Dirichlet functional. In Section 4 we introduce a class of related functionals, and in Section 5 we discuss local stability in this more general context. In Section 6 we discuss the Glansdorff-Prigogine stability criterion as a special case. Finally, in Section 7 we formulate a variational procedure based on the functionals introduced in Section 4.

2. Nonlinear Diffusion

The nonlinear diffusion equation for particles diffusing in a uniform medium is given by

$$\frac{\partial n}{\partial t} = \nabla \cdot [D(n) \nabla n]$$

(2.1)

where \(n(r, t)\) is the particle density and the diffusion coefficient \(D(n)\) is a positive function of the local density. The equation originates from the balance equation expressing conservation of particles

$$\frac{\partial n}{\partial t} + \nabla \cdot j = 0$$

(2.2)
and a constitutive equation for the current density \( j(r, t) \)

\[ j = -D(n) \nabla n. \]  

(2.3)

The particle density is easily accessible to measurement, but in a thermodynamic context the more natural variable to use would be the chemical potential \( \mu(r, t) \). Then the constitutive equation (2.3) is replaced by

\[ j = -\sigma(\mu) \nabla \mu \]  

(2.4)

where the conductivity \( \sigma \) is related to the diffusion coefficient \( D \) by

\[ D = (\frac{\partial \mu}{\partial n}) \sigma \]  

(2.5)

with the thermodynamic derivative \( (\partial \mu/\partial n) \) calculated from the equation of state \( \mu = \mu(n) \).

The nonlinear diffusion equation also appears in many problems of heat conduction. In this case the basic balance equation is

\[ \partial e/\partial t + \mathbf{V} \cdot \mathbf{q} = 0 \]  

(2.6)

where \( e(r, t) \) is the energy density, and \( \mathbf{q}(r, t) \) the heat current which is related to the temperature gradient by the constitutive equation

\[ \mathbf{q} = -\lambda(T) \nabla T, \]  

(2.7)

where \( \lambda(T) \) is the heat conductivity.

Energy density and temperature are related by a thermodynamic equation of state \( e = e(T) \). Clearly Equations (2.2) and (2.4) are mathematically equivalent to Equations (2.6) and (2.7). In the sequel we shall use the language of particle diffusion.

The use of the variables \( \mu \) and \( T \) presupposes the possibility of describing the non-equilibrium state with the aid of thermodynamic state variables. Alternatively one may regard (2.1), or its equivalent in other contexts, as a phenomenological equation which may be valid independent of a thermodynamic description. Our mathematical discussion will be based on (2.1). From a mathematical point of view any choice of new variable is allowed and may be useful. A particularly effective choice of variable is

\[ \psi(n) = \int_{0}^{n} D(n') \, dn'. \]  

(2.8)

Since \( D(n) \) is positive, \( \psi(n) \) is a monotonically increasing function of \( n \) and one can invert to \( n = n(\psi) \). The equation for time-independent solutions \( \psi_0(r) \) becomes simply Laplace's equation

\[ \nabla^2 \psi_0 = 0 \]  

(2.9)

with a unique solution for suitable boundary conditions. The above transformation has already been used by Kirchhoff [7]. We show in the next section that any initial distribution \( n(r, 0) \) tends to the stationary distribution \( n_0(r) \) corresponding to \( \psi_0(r) \), as time goes on.

### 3. Dynamical Stability

The uniqueness of the steady state solution \( \psi_0(r) \) is proved by application of Dirichlet's principle. This states that the solution of the Laplace equation \( \nabla^2 \psi_0 = 0 \) in a region \( \Omega \) for given values of \( \psi \) or of \( n \cdot \nabla \psi \) on the boundary \( \Sigma \) is given by the function \( \psi(r) \) which minimizes the functional

\[ U(\psi) = \int_{\Omega} [V \psi]^2 \, d\mathbf{r} \]  

(3.1)

in the class of functions which are twice differentiable in \( \Omega \) and satisfy the boundary condition. Suppose there are two solutions \( \psi_{01} \) and \( \psi_{02} \), then the difference \( \psi_{01} - \psi_{02} \) again satisfies the Laplace equation with \( \psi_{01} - \psi_{02} \) or \( n \cdot \mathbf{V} (\psi_{01} - \psi_{02}) \) vanishing on \( \Sigma \). Since \( U(\psi) \) is positive semi-definite, \( \psi_{01} \) must equal \( \psi_{02} \), or can at most differ by a constant for the second boundary condition.

To prove dynamical stability we write

\[ n = n_0 + n_1, \quad \psi = \psi_0 + \psi_1, \]  

(3.2)

and observe that

\[ U(\psi) = \int_{\Omega} [(V \psi_0)^2 + (V \psi_1)^2] \, d\mathbf{r} = U(\psi_0) + U(\psi_1) \]  

(3.3)

where we have used that either \( \psi_1 \) or \( n \cdot \nabla \psi_0 \) vanishes on \( \Sigma \), and \( \nabla^2 \psi_0 = 0 \) in \( \Omega \). Deviations \( n_1 \) from the stationary distribution develop in time according to

\[ \frac{\partial n_1}{\partial t} + \mathbf{V} \cdot \mathbf{j}_1 = 0, \]  

(3.4)

where \( \mathbf{j}_1 = \mathbf{j}(n_0 + n_1) - \mathbf{j}(n_0) \). We construct the functional

\[ F(n, n_0) = \int_{\Omega} f(n, n_0) \, d\mathbf{r} \]  

(3.5)

where

\[ f(n, n_0) = \int_{n_0}^{n} \left[ \psi'(n') - \psi(n_0) \right] \, dn' = \int_{n_0}^{n} \psi_1(n', n_0) \, dn'. \]  

(3.6)

From (2.8) it follows that

\[ f(n, n_0) = \int_{n_0}^{n} \int_{n_0}^{n'} D(n'') \, dn'' \, dn' \]  

(3.7)