Low-Energy Scattering by Singular Potentials in the Linearized Approximation to Calogero’s Equation

H. Krüger
Physikalisches Institut der Universität Freiburg i. Br.

Eingegangen am 19. Juni 1967

The linearized Calogero equation for \( \tan \delta(r) \) has never been applied to singular potentials with a repulsive core. It is shown that the solution can be written in the form of integrals. Even in the low-energy limit rough estimates can at least be obtained. This can be seen from a comparison of scattering-lengths calculated, both exactly and approximately, for negative-power potentials.

The problem of calculating scattering amplitudes in potential fields with singularities has always met great difficulties. The Born approximation, e.g., leads to divergent integrals in such cases. It seems that this somewhat awkward situation has, to a certain extent, been remedied by Calogero’s¹ work, but the usual procedure of solving his differential equation for the interpolating phase shifts by successive iterations still does not succeed in the case of negative-power repulsion. The linearized approach to his differential equation, however, shows more to advantage for singular potentials. In this paper the exact solution of this approximate equation has been given from which not only can scattering lengths in the low energy approach be obtained, the integrals determining them no longer diverging, but also may the values be used as rough estimates and are in several cases even lying rather close to the exact results.

We define the interpolating tangent \( t_1(r) \), by putting the radial wave function for a given \( l \),

\[
\chi_l(r) = c_l(r) \left[ j_l(kr) - n_l(kr)t_l(r) \right].
\]

Here \( j_l(z) \) and \( n_l(z) \) are Riccati-Bessel functions defined in ². Introducing (1) into the radial wave equation

\[
\chi'' + \left[ k^2 - \frac{l(l+1)}{r^2} - W(r) \right] \chi_l = 0; \quad \chi_l(0) = 0
\]

and imposing upon the two functions \( c_l(r) \) and \( t_l(r) \) the condition

\[
\frac{c_l'}{c_l} (j_l - n_l t_l) = n_l t_l'
\]

in order to avoid second derivatives when putting (1) into (2), we obtain

\[ t'(r) = -\frac{W(r)}{k} \left[ j_i(kr) - n_i(kr) t_i(r) \right] \; ; \; \; \; t_i(0) = 0. \tag{4} \]

Eq. (4) is Calogero's differential equation for the interpolating tangent, the finite limit

\[ t_i(\infty) = \tan \delta_i(k) \tag{5} \]

determining the phase shift \( \delta_i \) of the wave function. It is useful for what follows to mention that from (1) and (3) then results by a quite simple calculation that

\[ \frac{j_i'}{j_i} = \frac{j_i'}{j_i} - n_i \frac{t_i}{j_i}, \quad \text{resp.} \quad t_i(r) = \frac{j_i}{j_i} \frac{j_i'}{j_i} - n_i \frac{t_i}{j_i} \tag{6} \]

The basic equation to our approximate solution is obtained from (4) by neglecting the \( t^2 \)-term. Thus we arrive at the linearized equation

\[ \tilde{t}'(r) = -\frac{W(r)}{k} \left[ j_i^2(kr) - 2j_i(kr) n_i(kr) \tilde{t}_i(r) \right] \; ; \; \; \tilde{t}_i(0) = 0 \tag{7} \]

for the approximation \( \tilde{t}_i \) to \( t_i \) which permits exact integration and yields the result

\[ \tilde{t}_i(\infty) = \tan \tilde{\delta}_i \]

\[ = -\frac{1}{k} \int_0^\infty dr W(r) j_i^2(kr) \exp \left\{ -\frac{2}{k} \int_0^\infty dr' W(r') j_i(kr') n_i(kr') \right\}. \tag{8} \]

This formula has so far never been applied to singular potentials with a repulsive core. It seems remarkable that even in that case the integral (8) still converges, since then the exponential goes rapidly to zero for small \( r \). It therefore is interesting to investigate whether the approximate value \( \tilde{t}_i(\infty) \) thus determined is a reliable approach to the exact value \( t_i(\infty) \), respectively to find in which energy interval the approximation may be used to advantage.

At low energies the validity of the approximation may be tested by expanding into powers of \( k \). As is well known, the \( l \)-th scattering length, \( a_t \) defined by

\[ a_t = -\lim_{k \to 0} \left\{ k^{-(2l+1)} \tan \delta_i(k) \right\} \tag{9} \]

is the first term in such an expansion. In the approximation (8) we find instead (replacing \( j_i \) and \( n_i \) by the lowest power of their respective series

\[ a_t = a_t^2 \int_0^\infty dr r^{2l+2} W(r) \exp \left\{ -\frac{2}{2l+1} \int_0^\infty dr' W(r') r' \right\}, \tag{10} \]

with \( a_t = \frac{2^l l!}{(2l+1)!} \).