INVARIANT SUBSPACES OF WEIGHTED SHIFT OPERATORS

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Let $s$ be a weighted shift operator in $l^p$, $p \in [1, \infty)$: $s(\xi_0, \xi_1, \ldots) = (0, \lambda_1 \xi_1, \lambda_2 \xi_2, \ldots)$. One proves its unicellularity under the condition $|\lambda_i| \not\to 0$ and also under some weaker conditions. One obtains also unicellularity conditions for weighted shift operators in Banach spaces of numerical sequences. One gives a new proof of the following theorem of M. P. Thomas: if $(\prod_{i=0}^{n-1} |\lambda_i| / \lambda_n) / \lambda_0 \not\to 0$ and $|\lambda_i| = O(i^{-e})$, $e > 0$, then the operator $s$ is unicellular in $l^p$. One considers also a multiple weighted shift, corresponding to the case when $\lambda_i$ are finite-dimensional vectors. Under the condition $\sum_{i=0}^{n} |\lambda_i| \not\to 0$, $\mu_n \not\to 0$ one obtains the description of the invariant subspaces of this operator, using formal matrix power series.

This paper is devoted to the problem of the description of (closed) invariant subspaces of the weighted shift operator:

$s(\beta_0, \beta_1, \ldots) = (0, \lambda_1 \beta_1, \lambda_2 \beta_2, \ldots)$. These operators are considered in the spaces $l^p$, $1 < p < \infty$ and in some other spaces (in particular, in spaces of vector sequences), while the "weights" $\lambda_i$ are subjected to certain "regularity" conditions of decrease as $i \to \infty$.

After the first investigations of W. F. Donoghue, Jr. (1964) and N. K. Nikol'skii (1965), several publications have appeared on weighted shift operators. The approach, connected with the passage to the space of formal power series, in which the operator $S$ acquires the form of a multiplication by an independent variable, has turned out to be convenient. Under certain additional assumptions on the weights $\lambda_i$, the obtained space of formal power series turns out to be a Banach algebra relative to multiplication and this allows us to describe the cyclic vectors of the operator $S$.

We recall that a collection of vectors $x^1, \ldots, x^q$ is said to be a cyclic collection for an operator $S$, acting in the space $K$, if $K = \operatorname{span}(x^1, \ldots, x^q)$, here

$$\operatorname{span}(\cdot) = \operatorname{span}(\cdot)$$

Here $\operatorname{span}(\cdot)$ is the closed linear span of the set $(\cdot)$. An operator $S$ is said to be unicellular [1] if the set of its invariant subspaces is linearly ordered by inclusion.

N. K. Nikol'skii has established [2] with the use of the Gelfand theory of commutative Banach algebras that a weighted shift operator $S$ is unicellular in a large class of cases when the sequence $|\lambda_i|_{i \geq 0}$ tends "regularly" to zero. In [2] one has obtained also results which show that the unicellularity of the operator $S$ holds if $|\lambda_i|_{i \geq 0}$ and if $|\lambda_i|_{i \geq m} = O(i^e)$ for some $m \in \mathbb{N}$ and some $i \in \mathbb{N}$ and $e > 0$. N. K. Nikol'skii has given examples of nonunicellular weighted shift operators in $l^p$, $p > 1$, with weights going nonmonotonically to zero (see [2]).

In connection with this, N. K. Nikol'skii [2, p. 190] and A. L. Shields ([3], Question 19) have posed the following question: is the shift operator unicellular in $l^p$, $p > 1$, under the unique assumption that $|\lambda_i| \not\to 0$, which does not allow the use of Banach algebra techniques. A. L. Shields has also formulated two weaker assertions, whose validity was unknown. Theorem 3, proved below, gives a positive answer to the mentioned question. Moreover, from Theorem 4 it follows that for the unicellularity of the shift operator it is sufficient to have $\lim_{i \to \infty} |\lambda_i| = 0$ and $\exists \mu_i |\lambda_i|_{i \geq m} = O(i^e)$ and even the "monotonicity in the mean" $\lim_{m \to \infty} |\lambda_i|_{i \geq m} = 0$ as $i \to \infty$.

The method of proof of these statements evokes the algebraic approach of [2] (see Theorem 1 of Sec. 1 of the present paper). Namely, for the proof of the cyclicity of a vector $b$

with \( b_0 \neq 0 \) we consider the formal power series \( x(\xi) = \sum_{n=0}^{\infty} (\lambda_0 \ldots \lambda_{n-1})^{-1} b_n \xi^n \) and its (formal) inverse series \( g(\xi) = \sum_{n=0}^{\infty} g_n \xi^n : x(\xi) g(\xi) \equiv 1 \). After suitable estimates of the coefficients \( g_n \) (which form the fundamental part of the paper) it turns out that

\[
\lim_{N \to \infty} \left( \frac{1}{N} \sum_{k=0}^{N-1} g_k \xi^k \right) = 0, \quad \text{where} \quad [g_k]_N(S) = \sum_{n=0}^{N} g_n \xi^n,
\]

from where one obtains the cyclicity of the vector \( b \).

The paper consists of five sections.

In Sec. 1 we describe the method presented in [2] and we prove a statement regarding the structure of the set of cyclic vectors (Theorem 2).

Section 2 is devoted to the formulation of the fundamental results on the weighted shift operator in \( L^p \). Also there one considers weighted shift operators in Banach spaces, related with general sequences of vectors \( \{e_n\}_{n=0}^{\infty} \) (Se \( n = \sum_{n=0}^{\infty} e_n e_{n+1} \)). Examples for the corresponding theorems are the weighted shift operators in the spaces \( L^p \) and \( C_A \). The fundamental results of Sec. 2 are formulated in [4].

Section 3 contains the proof of the theorems formulated in Sec. 2.

In Sec. 4, the technique developed in Sec. 3 is applied for the discovery of a new proof of a recent result of M. P. Thomas [5, 11] regarding the unicellularity of the shift operators in the spaces \( L^p \). The sufficient condition of Thomas

\[
\left( \prod_{k=0}^{I} \lambda_k \right)^{1/\xi} + 0, \quad \lambda_i = 0 \left( \frac{1}{\xi^I} \right), \xi > 0
\]

and the theorems of Sec. 2 do not overlap.

In Sec. 5 we consider the problem of the description of the invariant subspaces of the multiple weighted shift

\[
S(z_0, z_1, z_2, \ldots) = (0, \lambda_0 z_0, \lambda_1 z_1, \ldots),
\]

defined on the space of vector sequences \( L^p(F) = \{ z_i : z_i \in F, \sum_i \| z_i \|_F < \infty \} \) where \( \dim F < \infty \), while \( \lambda_i \) are linear operators in \( F \). The form of the results of this section has been suggested by [6].

In this case, a proper analogue of the theorems from Secs. 2, 3 can be considered the theorem on the description of the \( S \)-invariant subspaces under the condition of a certain "behavior regularity" of the spectra \( \sigma(\lambda_i), i \to \infty \). However, advanced results (the description of cyclic collections or invariant subspaces) can be obtained by requiring a "regular behavior" of the upper and lower bounds of the operators \( \lambda_i \).

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1. Algebras Related to a Shift Operator

Let \( \{ \lambda_i \}_{i=0}^{\infty} \) be a bounded sequence of nonzero complex numbers. We consider the weighted shift operator \( S \) in the space \( L^p \), \( 1 \leq p < \infty \), acting according to the rule

\[
(0, \lambda_0, \lambda_1 b_1, \lambda_2 b_2, \ldots),
\]

where \( \{ b_i \}_{i=0}^{\infty} \in L^p \). For \( p = \infty \) we assume that the operator \( S \) acts in the space \( c_0 \).

Let \( \omega = \{ \omega_n \}_{n=0}^{\infty} \) be a numerical sequence. We introduce the weighted spaces \( L^p(\omega) \) of numerical sequences \( x = \{ x_n \}_{n=0}^{\infty} \) with the finite norm

\[
\| x \|_{L^p(\omega)} = \left( \sum_{n=0}^{\infty} |x_n \omega_n|^p \right)^{1/p}, \quad 1 \leq p < \infty.
\]

for \( p = \infty \) we consider the space \( c_0(\omega) = \{ x : \lim \sup nx_n \omega_n = 0 \} \) with the norm \( \sup |x_n \omega_n| \).

**Lemma 1.** A weighted shift operator is unitarily equivalent to a shift \( S \) in the weighted space \( L^p(\omega) \), acting according to the rule