Traces of Functions from $H^\infty(\mathbb{B}^n)$ on Certain Sets of Hyperplanes

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One considers necessary and sufficient conditions in order that a collection of sections of a ball in $\mathbb{C}^n$ by hyperplanes be the set of zeros or the interpolation set for functions of class $H^\infty$ in the ball.

Let $\mathbb{B}^n = \{ z \in \mathbb{C}^n : |z| < 1 \}$ be the unit ball in the space $\mathbb{C}^n$, $n > 1$, let $H^\infty(\mathbb{B}^n)$ be the space of all bounded functions that are holomorphic in $\mathbb{B}^n$. For $a \in \mathbb{B}^n$, we define the hyperplane $T_a = \{ z \in \mathbb{C}^n : (z, a) = |a|^2 \}$ where $(z, a)$ is the usual inner product in $\mathbb{C}^n$. For a set $A \subseteq \mathbb{B}^n$ which is at most countable, we define the collection of sections $T_A = \bigcup_{a \in A} (T_a \cap \mathbb{B}^n)$. The following questions are natural:

1. What are the necessary and sufficient conditions on the set $A$ in order that there exist a function $f \in H^\infty(\mathbb{B}^n)$ such that $f^{-1}(A) = T_A$?

2. What are the necessary and sufficient conditions on the set $A$ in order that for each collection $\{ T_a \}_{a \in A}$ of functions $T_a \in H^\infty(\mathbb{B}^n)$, $H^\infty(\mathbb{B}^n) \ni f$ there should exist a function $f \in H^\infty(\mathbb{B}^n)$, such that $f|_{T_a \cap \mathbb{B}^n} = f_a$?

In the search for an answer to these questions, it seems appropriate to investigate also simpler situations, namely when the set $A$ is contained in a "sufficiently compact" part of the ball $\mathbb{B}^n$; this is undertaken in this note.

From the G. M. Khenkin-H. Skoda theorem [2] on the divisors of the functions of Nevanlinna class, there follows that if $f \in H^\infty(\mathbb{B}^n)$ and $T_A = f^{-1}(0)$, then we have

$$\sum_{a \in A} (1-|a|^2) < \infty.$$  \hspace{1cm} (1)

As shown by A. B. Aleksandrov [3], the fact that a function belongs to the space $H^\infty(\mathbb{B}^n)$ does not impose on the quantity $|a|$ any restriction in comparison to the Nevanlinna class, except (1): for each sequence $s_k > 0$ for which $\sum (1-s_k)^k < \infty$ there exists a function $f \in H^\infty(\mathbb{B}^n)$ such that $f^{-1}(0) = T_A$, $A = \bigcup \{ a_k \}$ and $|a_k| = s_k$. The difference between the zeros of the functions from $H^\infty(\mathbb{B}^n)$ and $N(\mathbb{B}^n)$ (the Nevanlinna class) consists in the fact the first ones have to be distributed "more uniformly" in $\mathbb{B}^n$, as shown by the following Theorem 1 (we recall that, by the G. M. Khenkin-H. Skoda theorem [2], for any set $A$, satisfying condition (1), there exists a function $f \in N(\mathbb{B}^n)$, such that $f^{-1}(0) = T_A$).

Definition. Let $\xi_0 \in \partial \mathbb{B}^n$, $0 < q < 1$, $\delta > 0$, $\zeta > 0$. By a $q$-wedge we mean the set $\mathcal{E}(q, \delta, \zeta) = \{ \xi \in \partial \mathbb{B}^n : \text{Im} \{ 1-|(\zeta, \xi)|^q \} = \sum_{\xi_0} \{ 1-|(\zeta, \xi)|^q \} < \frac{\zeta}{q} \}$.

The collection of sets $\mathcal{E}(q, \delta, \zeta)$, as $q$, $\delta$, and $\zeta$ vary, is equivalent in the sense of inclusion to the collection of Koranyi-Stein wedges

$$\mathcal{D}(\delta, \zeta) = \{ \xi \in \partial \mathbb{B}^n : \text{Re} \{ 1-|\xi|^q \} < \frac{\zeta}{q} \},$$

but here it is more convenient.

THEOREM 1. Let $f \in H^\alpha(B^n)$, $f \neq 0$, $A \subset E_{\xi}(\dot{\alpha}, a_0(\xi))$ for some $\eta, 0 < \eta < 1$, $\xi \in \partial B^n$, and let $T_A \subset \{f\}$. Then

$$\sum_{a \in A} (1-|a|)^{\xi} < c,$$

(2)

where

$$\xi = \begin{cases} 1, & 0 < \eta < \frac{1}{2}, \\ \alpha + \epsilon, & 0 < \eta = \frac{1}{2}, \\ \alpha + \frac{\epsilon}{\alpha} \ln \left(\frac{1}{\eta}\right), & 0 < \eta = \frac{1}{2}. \end{cases}$$

and the constant $c$ depends on $f$, $\eta$, $c_0$, $\delta$, $\epsilon$, but does not depend on the point $\xi \in \partial B^n$.

Remark. Let $B(\xi, \delta) = \{x \in B^n : |(\xi, \delta)| < \delta\}$, $\xi \in \partial B^n$. Integrating (2) with respect to $\xi$ for some $\eta$, $0 < \eta < (1/2)$, we obtain one more necessary condition in order that the function $f$ belong to the space $H^\alpha(B^n)$:

$$\sum_{a \in A} (1-|a|)^{\xi} < c,$$

(3)

and $c$ does not depend on $\xi$ and $\delta$. We note that for $f \in N(B^n)$ in the right-hand side of (3) we can set only $0(\delta)$.

The proof of Theorem 1 (under the assumption $\xi = (1, 0, 0, \ldots)$) is based on the investigation of the intersection of the set $T_A$ with a finite collection of analytic films $\Gamma_j$, defined by the equations

$$\xi = \sqrt{(1-\beta)}, \beta_j \in C^{n-1}, \xi = (\xi_j, \beta_j), \xi_j \in C^{n-1},$$

whose number depends on $\eta$ and $\epsilon$ if $q > 1/2$. Moreover, $|\beta_j| < \frac{1}{2} |1-\xi_j|$ if $q > 1/2$, and $B^n$ contains part of $\Gamma_j$ for which $\xi_j$ lies in the domain

$$G_j = \{x \in C : |x| + |\beta_j| + |1-\xi_j| < 1\}.$$

Moreover, if $(\xi_{aj}, \beta_{aj})$ is a point of intersection of $T_A$ and $\Gamma_j$, then for some subset $A_j \subset A$ one has

$$|1-|\xi_{aj}| - (1-|a|), a \in A_j,$$

(4)

with $\bigcup_{a \in A_j} = A$, and the points $\xi_{aj}$ lie in a smaller angle than the angle $\beta G_j$ at the point 1. Since the function

$$t_j(\xi) = \left(\frac{\xi_j}{\sqrt{1-\xi_j}} - \beta_j\right),$$

belongs to $H^\alpha(G_j)$ and $t_j(\xi_{aj}) = 0$, $a \in A_j$, mapping $G_j$ onto the circle $D$ and taking into account (4), we obtain (2).

2. For the interpolational sets $T_A$ in $H^\alpha(B^n)$ the natural necessary condition holds [1]; in [4] Amar has given a sufficient condition (condition (US)) in order that the set $T_A$ be interpolational in $H^\alpha(B^n)$, $4 \eta < \infty$, and also for the fact that from the given interpolation $f_A \in H^\alpha(T_A \cap B^n)$ one should find $\eta \in \beta MOA(B^n)$ (see the definition in [4]) such that $\eta |_{T_A \cap B^n} = \eta_A$.

In the case when $A \subset E_{\xi}(\dot{\alpha})$, $0 < \eta < 1$, one has a sufficient interpolation condition for $T_A$ in $H^\alpha(B^n)$, which follows from Amar's condition (US).

THEOREM 2. Let $A \subset E_{\xi}(\dot{\alpha})$, $0 < \eta < 1$, $\xi \in \partial B^n$ and assume that there exists $\delta_0 > 0$ such that

$$\inf_{x \in T_{\alpha_0} \cap B^n} \left(\frac{|(\xi_0, a_0) + (a_0, a_0)|}{1-|\xi_0|}\right) \geq \delta_0,$$

(5)

for any $a_0, a_0 \in A, a_0 \neq a_0$. Then for any collection of functions $f_A \in H^\alpha(T_A \cap B^n)$, $a \in A, f_{\dot{A}} \in H^\alpha(T_{\dot{A}} \cap B^n)$.