TRACES OF FUNCTIONS FROM $H^\infty(\mathbb{B}^n)$ ON CERTAIN SETS OF HYPERPLANES

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One considers necessary and sufficient conditions in order that a collection of sections of a ball in $\mathbb{C}^n$ by hyperplanes be the set of zeros or the interpolation set for functions of class $H^\infty$ in the ball.

Let $B^n = \{ z \in \mathbb{C}^n : |z| < 1 \}$ be the unit ball in the space $\mathbb{C}^n$, $n > 1$, let $H^\infty(\mathbb{B}^n)$ be the space of all bounded functions that are holomorphic in $\mathbb{B}^n$. For $a \in \mathbb{B}^n$ we define the hyperplane $T_a = \{ z \in \mathbb{C}^n : (z, a) = |a|^2 \}$ where $(z, a)$ is the usual inner product in $\mathbb{C}^n$. For a set $A \subset \mathbb{B}^n$ which is at most countable, we define the collection of sections $T_A = \bigcup_{a \in A} (T_a \cap \mathbb{B}^n)$. The following questions are natural:

1. What are the necessary and sufficient conditions on the set $A$ in order that there exist a function $f \in H^\infty(\mathbb{B}^n)$ such that $f^{-1}(0) = T_A$?

2. What are the necessary and sufficient conditions on the set $A$ in order that for each collection $\{ f_a \}_{a \in A}$ of functions $f_a \in H^\infty(\mathbb{B}^n)$, $\| f_a \|_{H^\infty(\mathbb{B}^n)} < 1$ there should exist a function $f \in H^\infty(\mathbb{B}^n)$, such that $f|_{T_a \cap \mathbb{B}^n} = f_a$?

In the search for an answer to these questions, it seems appropriate to investigate also simpler situations, namely when the set $A$ is contained in a "sufficiently compact" part of the ball $\mathbb{B}$; this is undertaken in this note.

From the G. M. Khenkin-H. Skoda theorem [2] on the divisors of the functions of Nevanlinna class, there follows that if $f \in H^\infty(\mathbb{B}^n)$ and $T_A = f^{-1}(0)$, then we have

$$\sum_{a \in A} (1-|a|)^n < \infty.$$ (1)

As shown by A. B. Aleksandrov [3], the fact that a function belongs to the space $H^\infty(\mathbb{B}^n)$ does not impose on the quantity $|a|$ any restriction in comparison to the Nevanlinna class, except (1); for each sequence $s_k > 0$ for which $\sum (1-s_k)^n < \infty$ there exists a function $f \in H^\infty(\mathbb{B}^n)$ such that $f^{-1}(0) = T_A$, $A=\bigcup \{a_k\}$ and $|a_k| = s_k$. The difference between the zeros of the functions from $H^\infty(\mathbb{B}^n)$ and $N(\mathbb{B}^n)$ (the Nevanlinna class) consists in the fact the first ones have to be distributed "more uniformly" in $\mathbb{B}^n$, as shown by the following Theorem 1 (we recall that, by the G. M. Khenkin-H. Skoda theorem [2], for any set $A$, satisfying condition (1), there exists a function $f \in N(\mathbb{B}^n)$, such that $f^{-1}(0) = T_A$).

Definition. Let $\xi_n \in \partial \mathbb{B}^n$, $0 < q < 1$, $\delta > 0$, $c > 0$. By a $q$-wedge we mean the set

$$E_q(\xi_n) = E_{q, \xi_n, c}(\xi_n) = \{ z \in \mathbb{B}^n : \text{Im} (f(z, \xi_n) = \mathbb{C} \forall f(z, \xi_n)); \| f \| = q; \text{Re} (f(z, \xi_n)) \leq c; |z|^q - |z_n|^q \leq \delta \}.$$ (2)

The collection of sets $E_q(\xi_n)$, as $q$, $\delta$, and $c$ vary, is equivalent in the sense of inclusion to the collection of Koranyi–Stein wedges

$$O_d(\xi_n) = \{ z \in \mathbb{B}^n : |z|^q - |z_n|^q \leq \delta \},$$

but here it is more convenient.

THEOREM 1. Let \( f \in H^\omega(H^s), f \neq 0, A \subseteq E_q, a_1(\xi) \) for some \( q, \omega < q < \frac{1}{2}, \xi \in \partial B^n \), and let \( T_A = \{ x \} \).

Then

\[
\sum_{a \in A} (1-|a|)^{\omega} < c,
\]

where

\[
x = \begin{cases} 
4, & 0 < q < \frac{1}{2}, \\
4 + \varepsilon, & q = \frac{1}{2}, \\
\infty, & \text{ otherwise}, \end{cases}
\]

and the constant \( c \) depends on \( f, q, \omega, \delta, c \), but does not depend on the point \( \xi \in \partial B^n \).

Remark. Let \( B(\xi, \delta) = \{ x \in B^n : H(\xi, \delta) \} \), \( \xi \in \partial B^n \). Integrating (2) with respect to \( \xi \) for some \( q, 0 < q < (1/2) \), we obtain one more necessary condition in order that the function \( f \) belong to the space \( H^\omega(B^n) \):

\[
\sum_{a \in A} (1-|a|)^{\omega} < \delta c^n
\]

and \( c \) does not depend on \( \xi \) and \( \delta \). We note that for \( f \in N(B^n) \) in the right-hand side of (3) we can set only \( 0(\delta) \).

The proof of Theorem 1 (under the assumption \( \xi = (1, 0, 0, \ldots) \)) is based on the investigation of the intersection of the set \( T_A \) with a finite collection of analytic films \( \Gamma_j \), defined by the equations

\[
\xi' = \sqrt{-r^2} \cdot \gamma_j, \gamma_j \in \mathbb{C}^{n-1}, \xi = (\xi, \xi'), \xi \in \mathbb{C}^n,
\]

whose number depends on \( q \) and \( \varepsilon \) if \( q > 1/2 \). Moreover, \( |\gamma_j|^2 \leq \frac{1}{2} + \varepsilon(\xi) \) if \( q \geq 1/2 \), and \( B^n \) contains part of \( \Gamma_j \) for which \( \xi_j \) lies in the domain

\[
G_j = \{ x \in \mathbb{C} : |x| < \frac{1}{2} + |\gamma_j|^2 - |\xi_j| < 1 \}.
\]

Moreover, if \( (\xi_{aj}, \gamma_{aj}) \) is a point of intersection of \( T_A \) and \( \Gamma_j \), then for some subset \( A_j \subseteq A \) one has

\[
(1-|\xi_{aj}|)^{\omega} \leq (1-|a|)^{\omega}, \quad a \in A_j,
\]

with \( \bigcup A_j = A \), and the points \( \xi_{aj} \) lie in a smaller angle than the angle \( \theta G_j \) at the point 1. Since the function

\[
\frac{1}{\varepsilon} (\xi_j, \sqrt{-r^2} \cdot \gamma_j)
\]

belongs to \( H^\omega(G_j) \) and \( \frac{1}{\varepsilon} (\xi_{aj}, \sqrt{-r^2} \cdot \gamma_{aj}) = 0 \), \( a \in A_j \), mapping \( G_j \) onto the circle \( D \) and taking into account (4), we obtain (2).

2. For the interpolational sets \( T_A \) in \( H^\omega(B^n) \) the natural necessary condition holds [1]; in [4] Amar has given a sufficient condition (condition (US)) in order that the set \( T_A \) be interpolational in \( H^\omega(B^n) \), \( \omega < \infty \), and also for the fact that from the given interpolation \( \{ f \} \in H^\omega(T_A \cap B^n) \), one should find \( \eta \in BMOA(B^n) \) (see the definition in [4]) such that \( \eta|_{T_A \cap B^n} = \{ f \} \). In the case when \( A \subseteq E_q(\xi), 0 < q < 1 \), one has a sufficient interpolation condition for \( T_A \) in \( H^\omega(B^n) \), which follows from Amar's condition (US).

THEOREM 2. Let \( A \subseteq E_q(\xi), 0 < q < 1, \xi \in \partial B^n \) and assume that there exists \( \delta_0 > 0 \) such that

\[
\inf_{x \in T_A \cap B^n} \left| \frac{(x, a)}{1-(x, a)} \right| > \delta_0
\]

for any \( a \in A \), \( a \neq a' \). Then for any collection of functions \( \{ f \} \in H^\omega(T_A \cap B^n), a \in A, H^\omega(T_A \cap B^n) \)

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