ON THE DEVIATION AND THE TYPE OF A COHEN-MACaulay RING

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In this note we discuss the question, which pairs of integers \((d,r)\) can occur as the deviation \(d\) and the type \(r\) of a Cohen-Macaulay domain.

The type of a local Cohen-Macaulay ring \(R\) is the number

\[
r(R) := \dim_K \text{Ext}^d_R(K,R),
\]

where \(d = \text{dim } R\) and \(K\) is the residue field of \(R\). If \(\{x_1,...,x_d\}\) is a maximal regular sequence in the maximal ideal \(\mathfrak{m}\) of \(R\), then

\[
r(R) = \dim_K \text{Hom}_R(K,R/(x_1,...,x_d)),
\]

the dimension of the socle of \(R/(x_1,...,x_d)\) as a \(K\)-vector space. The following rules hold:

a) \(r(R) = r(R/(x_1,...,x_i))\) for each regular sequence \(\{x_1,...,x_i\}\) in \(\mathfrak{m}\).

b) If \(\varphi : R \rightarrow S\) is a local homomorphism such that \(S\) is a flat \(R\)-module, then \(S\) is Cohen-Macaulay iff \(R\) and \(S/\mathfrak{m}S\) are Cohen-Macaulay. In this case \(r(S) = r(R) \cdot r(S/\mathfrak{m}S)\) (see [2], 1.24).

Let \(K\) be an arbitrary field and let \(\text{CM}\) be the set of isomorphy classes of Cohen-Macaulay rings of the form
Let \( \mu(I) \) be the minimal number of generators of \( I \). The number \( d(R) := \mu(I) - (n - \dim R) \) is an invariant of \( R \), which is called the deviation of \( R \).

We have the following rule:

c) If \( \{x_1, \ldots, x_i\} \) is a regular sequence in the maximal ideal of \( R \), then \( d(R) = d(R/(x_1, \ldots, x_i)) \) (see [1], Lemma 1.2).

**PROPOSITION 1.** CM, with respect to complete tensor product, is a semigroup and the map

\[ \alpha : \text{CM} \rightarrow \mathbb{N}_0^+ \times \mathbb{N}^x, \alpha(R) = (d(R), r(R)) \]

is a semigroup homomorphism. (Here \( \mathbb{N}_0^+ \) is the additive semigroup of non-negative integers and \( \mathbb{N}^x \) the multiplicative semigroup of positive integers).

**PROOF.** Let \( R \) be given as above, \( \dim R = d \), and let \( S = K[[Y_1, \ldots, Y_{m'}]]/J \) be another ring in CM with \( \dim S = d' \). Then \( R \otimes_K S = K[[X_1, \ldots, X_n, Y_1, \ldots, Y_{m'}]]/(I, J) \) has dimension \( d + d' \) and obviously \( \mu((I,J)) = \mu(I) + \mu(J) \). From this follows \( d(R \otimes_K S) = d(R) + d(S) \).

The natural map \( R \rightarrow R \otimes_K S \) is flat, so by rule b) we see that \( R \otimes_K S \) is in CM and \( r(R \otimes_K S) = r(R) \cdot r(S) \).

The elements of CM mapped by \( \alpha \) onto the neutral element \((0,1)\) of \( \mathbb{N}_0^+ \times \mathbb{N}^x \) are the complete intersections. We are interested in the image of \( \alpha \):