On Functions and Measures whose Fourier Transforms are Functions

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1. Introduction

The theory of integral transformations in the form developed in [1] yields in the special case of the Fourier transformation some interesting information concerning classes of functions whose Fourier transforms are functions. Some results in this direction were outlined in [3] in the case of one dimensional Fourier transformation; we shall develop here the ideas of that paper in the case of the Fourier transformation on $\mathbb{R}^n$ and on some locally compact abelian groups.

To make this paper self-contained we have to begin with a brief outline of relevant concepts and results introduced in [1].

Let $(X, \mu)$, $(Y, \nu)$ be two totally $\sigma$-finite measure spaces and denote by $\mathcal{M} = \mathcal{M}(X, \mu)$, $\mathcal{N} = \mathcal{M}(Y, \nu)$ the linear spaces of (equivalence classes of) measurable finite almost everywhere complex valued functions on $X$ and $Y$, provided with the topologies of convergence in measure on all subsets of finite measure. $\mathcal{M}$ and $\mathcal{N}$ are complete linear metric spaces. It is convenient to choose once and for all some specific metrics defining the topologies in $\mathcal{M}$ and $\mathcal{N}$; we define the translation invariant metrics $q_x$, $q_y$ on $\mathcal{M}$ and $\mathcal{N}$ by the formulas

$$q_x(u) = \int_X \frac{|u(x)|}{1 + |u(x)|} \phi(x) \, d\mu(x), \quad q_y(v) = \int_Y \frac{|v(y)|}{1 + |v(y)|} \psi(y) \, d\nu(y), \quad u \in \mathcal{M}, \ v \in \mathcal{N},$$

where $\phi, \psi > 0$ are fixed functions, $\phi \in \mathcal{M}$, $\psi \in \mathcal{N}$ such that

$$\int_X \phi \, d\mu = \int_Y \psi \, d\nu = 1.$$

For a measurable complex valued function $K(x, y)$ on the product space $(X \times Y, \mu \times \nu)$ the integral transformation $K : \mathcal{M} \to \mathcal{N}$ with the kernel $K(x, y)$ is given by the formula

$$(Ku)(y) = \int_X K(x, y) u(x) \, d\mu(x).$$

$K$ is well defined on the linear subspace of $\mathcal{M}$ of all functions $u$ such that

$$\int_X |K(x, y)| |u(x)| \, d\mu(x) < \infty \quad \text{a.e.}$$

This subspace of $\mathcal{M}$ is referred to as the proper domain of $K$ and is denoted by $\mathcal{D}_K$.

We assume in what follows that $K$ is non singular, i.e. there exists $u \in \mathcal{D}_K$ such that $u \neq 0$ a.e.
We introduce in $\mathcal{M}$ a partial ordering setting $u \preceq v$ if and only if $|u(x)| \leq |v(x)|$ a.e.

If $A$ is a linear metric space of measurable functions continuously contained in $\mathcal{M}$ then $K$ is said to be $A$-semi regular ($A$-s.r.) if i) $\mathcal{D}_K \cap A$ is a dense subspace of $A$, ii) the transformation $K: \mathcal{D}_K \cap A \to \mathcal{M}$ is continuous if $\mathcal{D}_K \cap A$ is provided with the topology of $A$. If $K$ is $A$-s.r. then $K$ can be extended (by continuity) to a continuous linear transformation of $A$ into $\mathcal{M}$, which we shall denote by $K_A$ and refer to as $A$-extension of $K$.

$A$ is an $FL$-subspace of $\mathcal{M}$ if $A$ is a complete metric space continuously contained in $\mathcal{M}$ and the conditions $u \in A$, $v \in \mathcal{M}$, $v \preceq u$, imply $v \in A$.

The maximal $FL$-subspace of $\mathcal{M}$ to which a given integral transformation can be extended by continuity is obtained by closing the proper domain $\mathcal{D}_K$ of $K$ in $\mathcal{M}$ provided with the metric

$$
\hat{g}_K(u) = g_X(u) + \sup \{g_Y(Kv); v \in \mathcal{D}_K, v \preceq u\}.
$$

(4)

The relevant properties of this metric are given in the following.

**Theorem 1.** (i) $\hat{g}_K$ is a complete translation invariant metric on $\mathcal{M}$. (ii) The closure $\tilde{\mathcal{D}}_K$ of $\mathcal{D}_K$ in $\mathcal{M}$ with the metric $\hat{g}_K$ is an $FL$-subspace of $\mathcal{M}$. (iii) $K$ is $\tilde{\mathcal{D}}_K$-semi regular, $\tilde{\mathcal{D}}_K$ being provided with the metric $\hat{g}_K$. (iv) Denote by $\tilde{K}$ the $\tilde{\mathcal{D}}_K$-extension of $K$. If $A$ is an $FL$-subspace of $\mathcal{M}$ such that $K$ is $A$-s.r. then $A$ is continuously contained in $\tilde{\mathcal{D}}_K$ (with the metric $\hat{g}_K$) and $K_A$ is identical with the restriction of $\tilde{K}$ to $A$.

The space $\tilde{\mathcal{D}}_K$ is called the extended domain of $K$. The following is a necessary condition for a function $u \in \mathcal{M}$ to belong to $\tilde{\mathcal{D}}_K$.

**Proposition 1.** If $u \in \tilde{\mathcal{D}}_K$ and $X = \bigcup_{k=1}^{\infty} X_k$ is a partition of $X$ then for every sequence $\{u_k\} \subset \mathcal{D}_K$ the condition $u_k \preceq \chi_{X_k} u$, $k = 1, 2, ...$ implies

$$
\sum_{k=1}^{\infty} |(Ku_k)(y)|^2 < \infty \quad \text{for almost every } y.
$$

(5)

In the proposition above and in what follows $\chi_E$ denotes the characteristic function of the set $E$.

In the case when $X$ and $Y$ are topological spaces and $\mu, \nu$ are measures defined on $\sigma$-fields of Borel subsets of $X$ and $Y$ we have the following proposition.

**Proposition 2.** If $X$ is locally compact, $Y$ satisfies the Lindelöf condition and $K(x, y)$ is continuous on $X \times Y$ then for every $u \in \tilde{\mathcal{D}}_K$ and compact $C \subset X$ we have $\chi_C u \in \tilde{\mathcal{D}}_K$.

2. Spaces $\mathcal{B}^{(2)}(\mathbb{R}^n)$ and $\mathcal{L}^{(2)}(\mathbb{R}^n)$

Let $X$ be a set and $S$ be a $\sigma$-field of subsets of $X$. If $P = \{X_k\}_{k=1}^{\infty}$, $X_k \in S$ is a partition of $X$, $p$ is a number, $1 \leq p < \infty$ and $\mu$ is a complex valued measure on $S$ then we define

$$
\|\mu\|_{p,p} = \left[ \sum_{k=1}^{\infty} (|\mu| (X_k))^{p} \right]^{1/p}.
$$

(6)