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On Schwartz Spaces

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This paper concerns a certain class of locally convex topological vector spaces, defined and called Schwartz spaces by A. Grothendieck. In a general setting, one can consider Schwartz spaces either as a special case of those locally convex spaces whose bounded subsets are precompact, or as a more general case of nuclear spaces. Moreover Schwartz spaces and nuclear spaces can be defined in the same manner and, if one uses the notion of diametral dimension, most of the permanence properties of both types can be obtained simultaneously.

After establishing some terminology in § 1, we define and give several characterizations of Schwartz spaces in § 2. The most important one is (4), which plays a central role in the later developments, where we consider different types of convergence in the duals of Schwartz spaces. In § 3 we introduce co-Schwartz spaces in analogy with conuclear spaces. In the last section we apply our results to F- and DF-spaces, where Montel spaces and local convergence of a sequence of linear functionals play an important role. By means of counter examples we show that a duality theorem for Schwartz spaces, similar to the duality theorem for nuclear spaces ([4], 5.1.9), does not exist. In notation and terminology we follow [3].

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§ 1. Terminology

We denote by $\mathfrak{U}(E)$ the collection of all closed and absolutely convex neighbourhoods of the origin of a locally convex space E. For such a neighbourhood U we define E_u to be the quotient space $E/N(U)$ normed by $\|x(U)\| = p_u(x)$, where p_u is the semi norm of U and $N(U)$ its kernel. If $V \in \mathfrak{U}(E)$ and $V \subset U$, then there exists a canonical linear mapping K_U^V of E_V onto E_U defined by

$$K_U^V[x(V)] = x(U).$$

We denote by $\mathfrak{B}(E)$ the collection of all closed, absolutely convex and bounded subsets of E. For such a bounded subset A we define $E(A)$ to be the subspace $\bigcup_{n=1}^{\infty} nA$ of E, normed by A. If $B \in \mathfrak{B}(E)$ and $B \supset A$, then we denote by I_B^A the canonical imbedding of $E(A)$ into $E(B)$.

If $U \in \mathfrak{U}(E)$, the strong dual of the normed space E_u is the Banach space $E'(U^0)$. The adjoint of K_U^V is the imbedding $I_{V^0}^{U^0}$ of $E'(U^0)$ into $E'(V^0)$.

If $A \in \mathfrak{B}(E)$, the space E'_{A^0} is norm-isomorphic to a subspace of $[E(A)]'$ and the restriction of the adjoint of I_B^A to E'_{B^0} is the mapping $K_{A^0}^{B^0}$ of E'_{B^0} onto E'_{A^0}. Similarly, the normed space $E(A)$ is norm-isomorphic to a subspace of $(E'_{A^0})'$ and the restriction of the adjoint of $K_{A^0}^{B^0}$ to $E(A)$ is the imbedding I_B^A.

§ 2. Schwartz Spaces

A locally convex space E is said to be a *Schwartz space*, if for every neighbourhood $U \in \mathfrak{U}(E)$ there exists another neighbourhood $V \in \mathfrak{U}(E)$, $V \subset U$, such that the mapping K_U^V is precompact.

Using the fact that a linear mapping of a normed space into a normed space is precompact if and only if its adjoint is compact, we immediately get the following characterization of Schwartz spaces from the definition.

(1) *E is a Schwartz space if and only if for every neighbourhood $U \in \mathfrak{U}(E)$, there exists a neighbourhood $V \in \mathfrak{U}(E)$ such that U^0 is a compact subset of $E'(V^0)$.*

We give two other characterizations of Schwartz spaces which have connections with the diametral dimension of locally convex spaces.

(2) *E is a Schwartz space if and only if for every $U \in \mathfrak{U}(E)$ there exists $V \in \mathfrak{U}(E)$ such that the following condition holds:*

For every $\varepsilon > 0$ there exists a finite dimensional subspace L of E with

$$V \subset \varepsilon U + L .$$

This is actually a simple reformulation of the definition. The next characterization is the dual of (2) and it follows from (1) by taking polars (see [7]).

(3) *E is a Schwartz space if and only if for every $U \in \mathfrak{U}(E)$ there exists $V \in \mathfrak{U}(E)$ such that the following condition holds:*

For every $\varepsilon > 0$ there exists a finite codimensional subspace M of E with

$$V \cap M \subset \varepsilon U .$$

Using these two results one can obtain most of the known permanence properties of Schwartz spaces. We will not go into this here. (For permanence properties of Schwartz spaces see [2, 7].)

We say a sequence (u_n) in the dual E' of a locally convex space E *converges locally*, if there exists a neighbourhood V of E such that each u_n is in $E'(V^0)$ and (u_n) converges to an element of $E'(V^0)$ in the norm topology. It is easy to see that every locally convergent sequence of continuous linear functionals converges strongly to the same limit.

(4) *A locally convex space E is a Schwartz space if and only if for every neighbourhood $U \in \mathfrak{U}(E)$ there exists a sequence (u_n) in E', locally convergent to zero, such that the inequality*

$$p_U(x) \leqq \sup_n |\langle u_n, x\rangle| \tag{5}$$

holds for all x in E.

Proof. If E is a Schwartz space, then the polar U^0 of a neighbourhood $U \in \mathfrak{U}(E)$ is a compact subset of the Banach space $E'(V^0)$ for some neighbour-