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Duality and Intersection Theory in Complex Manifolds. I.

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Abstract

We introduce the concept of a twisting cochain and a twisted complex associated to a coherent sheaf. For sheaves of submanifolds these twisted complexes are used to construct on cochain level the Grothendieck theory of dual class and Gysin map. These explicit constructions give, for instance, a local formula for dual class of higher codimensional submanifolds. We prove a refined version of the Hirzebruch Riemann Roch using such local formulas. We also prove a theorem on when global analytic intersection classes can be computed from first order geometric data. This theory will be used to prove the Holomorphic Lefschetz formula (in Part II) and the Hirzebruch Riemann Roch for analytic coherent sheaves.

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0. Introduction

This is the first of two papers devoted to intersection and duality problems in complex manifolds (and algebraic varieties in characteristic zero). We develop a very concrete version, in the Čech context and at the cochain level, of Grothendieck's theory of the dual class of a submanifold, and to study conditions

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under which the product of the dual classes of two submanifolds can be computed from their first order behavior at their intersection. Our motivation for developing this theory is the proof of the Holomorphic Lefschetz Formula for higher dimensional fixed point sets[1] conjectured by Atiyah and Bott and proved in the special case of group actions by the Atiyah Singer Index Theory. The reader may find a short summary of the proof in the announcement [13].

Besides laying the foundations of the Letschetz formula it will be shown elsewhere that the connection cocycles in Sections 1 and 2 can be refined to construct from the local resolutions of a coherent sheaf its Chern classes as Čech cochains in Hodge cohomology. This coupled with a suitable generalization of Section 6 enables the basic view point which is developed here to cover the problem of the Rieman Roch theorem for a coherent sheaf on a compact complex manifold.

To explain our view point in more details, let X be a (compact) complex manifold and Y a complex submanifold of codimension r, then Y defines a current on X and the dual class of Y in X according to deRham [2] is simply a smoothing of this current. More generally, and it is most important for applications, we will consider currents supported on Y which act on forms with coefficients in holomorphic vector bundles E over X. The smoothing of this current then has coefficients in the adjoint bundle E' and we continue to call it a "dual class". Our first problem is to construct in E' valued Čech cochains a canonical representative of the dual class that depends only on the local geometry of the embedding of Y in X and transition functions of E. In the case of divisors of course such a formula is well known. For this purpose we may interpret Grothendieck's theory as that the space of $\bar{\partial}$-cohomology classes of δ-functions supported on Y (by δ-functions we mean currents supported on Y which do not involve derivatives) has an isomorphic description as the derived functor $\mathrm{Ext}^*(X;\mathcal{O}_Y,E')$. Then the "smoothing of δ-functions" becomes simply the natural transformation

$$\mathrm{Ext}^*(X;\mathcal{O}_Y,E')\to H^*(X,E') \tag{0.1}$$

induced by the projection $\mathcal{O}_X\to\mathcal{O}_Y$. [Here we use the fact that $H^*(X,E')\approx\mathrm{Ext}^*(X;\mathcal{O}_X,E')$.] The representative of the current of Y in $\mathrm{Ext}^r(X;\mathcal{O}_Y,E')$ is obtained by applying the inverse of the natural transformation

$$\mathrm{Ext}^r(X;\mathcal{O}_Y,E')\xrightarrow{\approx}H^0(X,\underline{\mathrm{Ext}}^r\,\mathcal{O}_X(\mathcal{O}_Y,E')) \tag{0.2}$$

to an obvious local smoothing of Y in the second group. The usual dual class construction is the case in which $E=\Omega_X^{n-r}$ so that $E'=\Omega_X^r$. Our problem is therefore to find a Čech description of Ext in which (0.1) and (0.2) are defined on cochains. Althigh this is possible when $\mathcal{O}_Y$ has a global resolution by vector bundles, these are not known to exist in general and further more it is not clear how the local data of Y is reflected in such resolutions in a canonical way. Fortunately there is a *locally functorial complex* of Čech type that computes Ext, namely the mysterious "twisted complex" whixh first occured in [12]. This complex arises from the local Koszul resolutions and a canonical sequence of

[1] The remaining details of proof is in Part II, the holomorphic Lefschetz formula to appear in Annals of Mathematics 1978