A Note on the kernel of the $\bar{\partial}$-Neumann operator
on strongly pseudo-convex domains
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Abstract
In this paper, we discuss the relations between a special Heisenberg coordinate system and a normalized Levi metric on strongly pseudo-convex domains in $\mathbb{C}^n$ and see how they are related to the $\bar{\partial}$-Neumann operator.

1. Introduction
The study of boundary regularity for solution of $\bar{\partial}$-Neumann problem

\begin{align}
\Delta u &= (\bar{\partial}^{\alpha^*} + \bar{\partial}^{\beta^*} )u = f \\
\bar{\partial} u &\in \text{domain (}\bar{\partial}^{\alpha^*}\text{)}, \quad \bar{\partial} u \in \text{domain (}\bar{\partial}^{\beta^*}\text{)}
\end{align}

on pseudo-convex domains $\Omega$ in $\mathbb{C}^{n+1}$ has been an interesting theme in the theory of several complex variables for many years. The existence and regularity properties of the $\bar{\partial}$-Neumann operator $N$ (the parametrix for this problem) on strongly pseudo-convex domains were well understood by the results of Kohn [6], Greiner-Stein [8] (for "Levi metric" case), Beals-Greiner-Stanton [2], and Chang [4] (for "non-Levi metric" cases).

On the other hand, it is very interesting to give a more explicit constructions of the operator $N$, which expresses the solution $u$ in terms of $f$. The result of the construction of $N$ were essentially achieved by two different methods. The first method (Phong [15] and Phong-Stein [16],[17]) involves techniques in partial differential equations; the second method (Lieb and Range [13],[14]) uses integral formulas.

Phong's result [15] is based on the point of view of the Dirichlet problem for the complex Laplacian on the "model" case $D=\{(z_1, z_2, \ldots, z_{n+1}) \in \mathbb{C}^{n+1} : \sum_{k=1}^{n+1} |z_k|^2 \}$ which was equipped with a Levi metric. He discovered that the kernel $N_j$ is a mixed type homogeneity kernel (mixed the Euclidean and the Heisenberg homogeneity). But unlike the parametrix of the boundary

(*) Work supported by MSRI, Berkeley, California.
complex Laplacian $\square_b$, Phong's result cannot be transferred to $\Omega$ directly by standard Heisenberg coordinates (we will explain this in section 2). After few years of effort, Phong and Stein [16],[17] achieved this goal by using special Heisenberg coordinates.

The method of Lieb and Range [12],[13],[14] is more fundamental, they used a generalized Bochner-Martinelli-Koppelman integral formula whose boundary term can be approximated with a Henkin type kernel. (Here the boundary geometry comes into play). In the papers [13] and [14], they found an integral operators $T_1$ and $(T_0)^*$ which are defined explicitly in terms of the geodesic distance function for the given Hermitian metric (the Euclidean homogeneity part) and the Levi polynomial of a plurisubharmonic defining function for $f_\Omega$ (the Heisenberg homogeneity part). $T_1$ and $(T_0)^*$ give the principal kernel to represent $\overline{\partial}^* N_1 f$ and $\partial N_0 f$ in terms of $f$. Under the assumption that $\overline{\Omega}$ is equipped with a "normalized Levi metric" (we will explain this in section 2), Lieb and Range solved the system of partial differential equations:

$$\overline{\partial} K_1 = \operatorname{ker}(T_1), \quad \overline{\partial}^* K_1 = \operatorname{ker}((T_0)^*)$$

(1.3)

to get the kernel $K_1$ for the $\overline{\partial}$-Neumann operator $N_1$.

It seems these two methods based on totally different philosophies. However, on the "model" $D$ a straightforward computation shows that Lieb-Range's kernel is equal to Phong's kernel plus an acceptable error term. How about the general domain $\Omega$? As we mentioned at the beginning, Phong-Stein need to use a special Heisenberg coordinates to transfer the kernel $N_1$ to $\Omega$. But Lieb-Range just use a standard Heisenberg coordinates to simplify the computations in their works (but they need to assume $\overline{\Omega}$ is equipped with a normalized Levi metric to construct the kernel $K$ for $N_1$). In this paper, we see how the special Heisenberg coordinates and a normalized Levi metric are related. The author would like to reiterate his thanks to his teacher and advisor E.M.Stein, not only for his many valuable suggestions but for his inspiring example in research and teaching. The author would also like to thank the referee for giving him some nice suggestions.

2. The special Heisenberg coordinates and a normalized Levi metric

Let us review some properties of strongly pseudo-convex domains. There are many equivalent definitions of strong pseudo-convexity, see Kerzman [9] and Krantz [11]. We will use the one which leads most quickly to the Heisenberg group (see [7],[20],[21] about the basic properties of the Heisenberg group):

(2.1) DEFINITION (Folland-Stein [7]):