Completion Problems for $j_{pq}$-Inner Functions. I.

D.Z. Arov, B. Fritzsche and B. Kirstein

This paper studies various completion problems for a subclass of $j_{pq}$-inner functions. Special attention is drawn to so-called $A$-normalized $j_{pq}$-elementary factors of full-rank, which are closely related to the matricial Schur problem. Finally, as an application an inverse problem for Carathéodory sequences is answered.

0. Introduction

The study of $J$-contractive matrix-valued functions has its origin in Potapov's famous paper [Pol]. In particular, he developed a factorization theory of such functions into a product of simplest factors of the same kind. What concerns the subclass of $J$-inner functions, the first author worked out a different concept of factorization, which has its roots in his former investigations on so-called generalized bitangential Schur-Nevanlinna-Pick interpolation problems (see [Aro3]-[Aro7]). Under some non-degeneracy condition the set of all solutions of such a problem can be parameterized with the aid of a linear fractional transformation of matrices generated by some $j_{pq}$-inner function which is built only from the original data of interpolation. Here $j_{pq}$ stands for the signature matrix $diag (I_p, -I_q)$. Conversely, there arises the question to describe the subset of all $j_{pq}$-inner functions $W$ for which there is a non-degenerate generalized bitangential Schur-Nevanlinna-Pick problem such that $W$ parameterizes its solution set in the above mentioned way. The study of this question led the first author to important subclasses of $j_{pq}$-inner functions, namely the sets of so-called regular and $A$-singular $j_{pq}$-inner functions. Modifying this problem one could ask which information contains a prescribed block of a
This paper is subdivided into two parts. The first one is aimed at the general theory of completing \( J_{pq} \)-inner functions. Particular attention will be paid to completion problems for \( J_{pq} \)-inner functions of Smirnov type and \( A \)-singular \( J_{pq} \)-inner functions. The second part of this paper handles completion problems for \( J_{pq} \)-inner polynomials with additional structure, namely for such matrix polynomials which can be considered as suitably normalized resolvent matrix of an appropriate non-degenerate matricial Schur problem. Hereby we will obtain a one-to-one correspondence between \( p \times q \) non-degenerate Schur sequences and \( p \times q \) matrix polynomials. This will yield a key for studying some inverse problem for matricial Carathéodory sequences, the scalar version of which was treated in [AA]. The investigations touch inverse problems for positive Hermitian block Toeplitz matrices, which, from its nature, are related to some topics considered in [Go]. One of the main themes of that book is the description of all non-singular Hermitian block Toeplitz matrices whose inverse has a prescribed first block column. In particular, I. Gohberg and L. Lerer studied very carefully this question (see [GL]). Using our investigations on completion of \( A \)-normalized full-rank \( J_{pq} \)-elementary factors we will uncover rather different correspondences between single block columns and positive Hermitian block Toeplitz matrices.

1. SOME BASIC FACTS ON VARIOUS CLASSES OF MEROMORPHIC MATRIX-VALUED FUNCTIONS IN THE UNIT DISC

In this first section, we will summarize some facts on particular classes of meromorphic functions. For a detailed treatment, we refer the reader to the monographs of Nevanlinna [Ne] and Duren [Dur]. We will start with some notations.

Throughout this paper, let \( p \) and \( q \) be positive integers. If \( m \) and \( n \) are non-negative integers with \( m \leq n \), then \( N^m_n \) stands for the set of all integers \( k \) with \( m \leq k \leq n \), whereas \( N_0^m \) designates the set of all non-negative integers. We will use \( \mathbb{C}, \mathbb{D}, \mathbb{T}, \hat{\mathbb{C}} \) and \( E \) to denote the set of