Math. Ann. 210, 83—104 (1974)

Equivariant Cohomology and Stable Cohomotopy

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1. Introduction

Equivariant cohomology theories have been defined axiomatically by various authors, notably Bredon [1] and tom Dieck [3]. In Section 2 we define what we call a G-cohomology theory using the equivariant analogs of the ordinary three axioms for a (non-equivariant) generalized cohomology theory. A novel feature of our definition (suggested by Graeme Segal) is that our theories are indexed by the elements $\alpha \in RO(G)$ – the real representation ring of some compact lie group G. This seems to be more natural than indexing by the integers.

If h_G is a G-cohomology theory then for each $\alpha \in RO(G)$ we have an equivariant cohomology theory in the sense of Bredon – namely $\{h_G^{\alpha+n}(\ ); n \in \mathbb{Z} \subset RO(G)\}$. In fact we may think of a G-cohomology theory as a collection $\{h_G^\alpha\}$ of Bredon theories – one for each $\alpha \in RO(G)$ – together with some compatibility conditions such as

$$\tilde{h}_G^\alpha(X) \cong \tilde{h}_G^{\alpha+V}(S^V \wedge X)$$

for every real G-module V, where $S^V = V^+$ the one-point compactification of V.

We define an "equivariant cohomology theory" as a collection $\{h_G\}$ of G-cohomology theories – one for each group G in some category of groups – together with some compatibility conditions. The most important condition being that there is a natural isomorphism

$$h_G^\alpha(G/H) \cong h_H^{\alpha|H}(pt)$$

if H is a subgroup of G. This definition avoids theories which are not really equivariant. An example of an equivariant cohomology theory is $\{K_G\}$ where G is a compact lie group.

Another, important, example is stable G-cohomotopy [14] denoted by $\Pi_{s,G}$ [1] – see Section 3. Its importance lies in the fact that it is a universal theory, (i.e. given any G-cohomology theory h_G, then there is a natural transformation $\Pi_{s,G} \to h_G$, etc.). We study $\Pi_{s,G}$ in Section 3, where we show, in particular, that $\Pi_{s,G}^0(pt)$ is the Burnside Ring if G is a finite group. Thus all G-cohomology theories (for a finite group G) are modules over the Burnside ring.

[1] Throughout this paper Π should be read as π.

In [7] we saw how to localize the Burnside ring at its various primes. In Section 4 we describe the result of localizing G-cohomology theories at the various primes $\mathscr{P}$ of the Burnside Ring $\Omega(G)$. We obtain a localization theorem which states that the map

$$h_G^\alpha(X) \to h_G^\alpha(X^{(A)})$$

induced by the inclusion $X^{(A)} \hookrightarrow X$, where $X^{(A)} = G \cdot X^A$, is an isomorphism when localized at the prime $\mathscr{P}(A, p)$ of $\Omega(G)$. Here p is a prime number or $p = 0$ and A is a subgroup of G containing no non-trivial p-quotient group. As a corollary we get

$$h_G^\alpha(X) \otimes \mathbf{Z}_{(p)} \cong \bigoplus h_G^\alpha(X^{(A)})_{\mathscr{P}(A,p)}$$

where the sum is taken over the conjugacy classes of the subgroups A of G which have no non-trivial p-quotient group.

In the special case of stable G-cohomotopy we can say more about $\Pi_{s,G}^\alpha(X) \otimes \mathbf{Z}_{(p)}$. For example, if p does not divide the order of G or $p = 0$ then there is an isomorphism

$$\Pi_{s,G}^n(X) \otimes \mathbf{Z}_{(p)} \cong \bigoplus \Pi_s^n(X^A/W(A)) \otimes \mathbf{Z}_{(p)}$$

for $n \in \mathbf{Z} \subset RO(G)$, where the sum is taken over the conjugacy classes of subgroups A of G and $W(A)$ denotes the quotient of the normalizer of A in G by A. See Theorem 4.10. If p does divide the order of the group then we have a corresponding result, so long as the group is abelian – see Theorem 4.15.

For the most part of this paper we have assumed that our groups are finite. In fact, from Definition 2.8 onwards G is a finite group, prior to that G is any compact lie group. In most cases it is not difficult to see the corresponding results and definitions if G is an arbitary compact lie group.

Throughout this paper we talk about cohomology theories, there are corresponding results and definitions for homology theories.

I would like to thank Graeme Segal very much indeed. Without his help and encouragement this paper may never have appeared.

2. Cohomology Theories

Let G be a compact lie group and $C_0(G)$ the category of compact G-spaces with base-point. A finite dimensional real vector space V on which G acts linearly will be called a G-module. It's one point compactification S^V, is a sphere with G action and is an object of $C_0(G)$ in which ∞ is regarded as base point. For each G-module V, there is a concept of *suspension*: if X is a compact G-space with base point x_0, define